COMPLEX ANALYSIS PRELIM EXAM. AUGUST 2020

$\mathbb{C}$ is the field of complex numbers $z = x + iy$, $\mathbb{R}$ the field of real numbers, $\mathbb{D}$ the open unit disk, and $\hat{\mathbb{C}}$ the Riemann sphere (aka, extended complex plane)

If you use a known theorem, or a similar result from a textbook, be sure to explicitly indicate so. Any other facts used must be proven. Proofs, or counter examples, are required for all problems.

(1) Consider the holomorphic map $\mathbb{C} \xrightarrow{\exp} \mathbb{C}$. Let L be an Euclidean straight line in $\mathbb{C}$.
   (a) Describe the preimage $\exp^{-1}(L)$ if $0 \in L$.
   (b) Suppose L lies in $\mathbb{C} \setminus \{0\}$. Pick $a \in L$ with $\operatorname{dist}(0, L) = |a|$. Describe the component of $\exp^{-1}(L)$ that contains the point $b := \operatorname{Log}(a)$. (Suggestion: Consider the cases where $a = 1, |a| = 1, a = re^{i\theta}$.)
   (c) Let $a \in L \subset \mathbb{C} \setminus \{0\}$ be as above. What can you say about $\exp^{-1}(L) \cap \exp^{-1}\big(\{ta \mid t \in \mathbb{R}\}\big)$?

(2) Evaluate $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2 + 1}\, dx.$

(3) Let f be a complex polynomial. Assume f has a simple zero at $z = a$.
   (a) Suppose $\Omega \xrightarrow{g} \mathbb{C}$ is holomorphic (in a domain Ω) and for each $z \in \Omega$, $g(z)^2 = f(z)$. Prove that $a \notin \Omega$.
   (b) Must the conclusion in part (a) hold if f has a non-simple zero at $z = a$?

(4) Let $u(x, y) = x\cos(y) + h(y)$ where h is a function of y alone. Prove that there is no holomorphic function f on the complex plane such that u is the real part of f.

(5) How many roots (counted with multiplicity) does the function $g(z) = 6z^3 + e^z + 1$ have in the unit disk $\mathbb{D} = \{z : |z| < 1\}$.