Topology Preliminary Examination August 2019
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Show all essential work.

1. Let $X$ be a compact metric space and $\{U_\alpha, \alpha \in A\}$ an open cover of $X$. Prove that there is a $\rho > 0$ such that if $d(x, y) < \rho$ then there exists an $\alpha \in A$ so that $x, y \in U_\alpha$.

2. Let $X$ be a space that is the union of subspaces $S_1, S_2, \ldots, S_n$, each of which is homeomorphic to the unit circle. Assume there is a point $p$ of $X$ such that $S_i \cap S_j = \{p\}$ for $i \neq j$.
   
   (a) Show that $X$ is Hausdorff if and only if each space $S_i$ is closed in $X$.
   
   (b) Give an example to show that $X$ need not to be Hausdorff.
   
   (c) Assume that $X$ is Hausdorff. Determine $\pi_1(X, p)$. Justify your answer.

3. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each $\alpha$.
   
   (a) Let $\prod X_\alpha$ have the product topology. Show that if $f$ is continuous then each function $f_\alpha$ is continuous.
   
   (b) Let $\prod X_\alpha$ have the product topology. Show that if each $f_\alpha$ is continuous then $f$ is continuous.
   
   (c) Let $\prod X_\alpha$ have the box topology. Give an example to show that if each $f_\alpha$ is continuous then $f$ need not be continuous.

4. Show that a path connected space is connected. Show that if $U \subset \mathbb{R}^n$ is open and connected then it is path connected.

5. Let $B \subset \mathbb{R}^3$ be the closed unit ball, so that

$$B = \{X = (x, y, z) : |X|^2 = x^2 + y^2 + z^2 \leq 1\}.$$  

Define an equivalence relation on $B$ by $X \sim Y$ if $|X| = |Y|$. Show that $B/\sim$ is homeomorphic to $[0, 1]$.

Recall that $X/\sim$ is the collection of equivalence classes determined by $\sim$. Introduce $q : X \to X/\sim$ as the function that assigns each point of $X$ to the equivalence class to which it belongs. The topology on $X/\sim$ is defined by saying $A \subset X/\sim$ is open if and only if $q^{-1}(A)$ is open in $X$, i.e., the union of the equivalence classes represented by the pts in $A$ is open in $X$.  
