Time allowed: 2 hours 30 minutes. 
Answer all problems and fully justify your work.

\( m^n \) is Lebesgue measure on \( \mathbb{R}^n \) and \( m^* \) is Lebesgue outer measure on \( \mathbb{R} \).

**Question 1**

Given any integrable function \( \varphi: [0, 1] \to \mathbb{R} \), define a corresponding sequence of functions by \( \varphi_n(x) := n \int_x^{x + \frac{1}{n}} \varphi(t) \, dt \) for \( x \in [0, 1] \) and \( n \in \mathbb{N} \). Here \( \varphi(t) := 0 \) for \( t \in (1, 2] \). With this definition, prove the following.

(1) If \( f, g: [0, 1] \to \mathbb{R} \) are integrable functions, then
\[
\int_0^1 |f_n(x) - g_n(x)| \, dx \leq \int_0^1 |f(x) - g(x)| \, dx.
\]

(2) If \( g: [0, 1] \to \mathbb{R} \) is continuous, then \( g_n \to g \) uniformly.

(3) If \( f: [0, 1] \to \mathbb{R} \) is integrable, then \( \int_0^1 |f_n - f| \, dt \to 0 \).

You may use without proof that for any \( \varepsilon > 0 \) there exists a continuous function \( g: [0, 1] \to \mathbb{R} \) such that \( \int_0^1 |f(x) - g(x)| \, dx < \varepsilon \).

**Question 2**

(1) Suppose \( f: \mathbb{R} \to \mathbb{R} \) satisfies \( |f(x) - f(y)| \leq L|x - y| \) for every \( x, y \in \mathbb{R} \). Prove that \( m^*(f(A)) \leq Lm^*(A) \) for every set \( A \subset \mathbb{R} \).

(2) Let \( f: \mathbb{R} \to \mathbb{R} \) be \( f(x) = e^{-x^2} \). Show that if \( A \subset \mathbb{R} \) with \( m(A) = 0 \) then \( m(f(A)) = 0 \).

(3) Suppose \( g: \mathbb{R} \to \mathbb{R} \) is a continuous function. For sets \( A \subset \mathbb{R} \), is it necessarily true that \( m(A) = 0 \) implies \( m(g(A)) = 0 \)?

Please turn over for remaining questions.
**Question 3**

Let \( f_n : [0, 1] \to \mathbb{R} \) be absolutely continuous functions with \( f_n(0) = 0 \). Suppose there is integrable \( g : [0, 1] \to \mathbb{R} \) such that \( \int_0^1 |f_n'(t) - g(t)| \, dt \to 0 \) as \( n \to \infty \).

Prove that \( f_n \) converge uniformly to some absolutely continuous function \( \phi : [0, 1] \to \mathbb{R} \) which satisfies \( \phi'(x) = g(x) \) for Lebesgue almost every \( x \).

**Question 4**

Let \( (X, \mathcal{F}, \mu) \) and \( (Y, \mathcal{G}, \nu) \) be measure spaces. Give the definition of the product measure \( \mu \times \nu \) and the family of sets on which it is defined.

Let \( L = ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1]) \subset \mathbb{R}^2 \) and \( \Delta = \{(x, x) : x \in \mathbb{R}\} \subset \mathbb{R}^2 \).

1. Find with proof the product measure \( (m \times m)(L) \).
2. Find with proof the product measure \( (m \times c)(L) \).
3. Find with proof the product measure \( (m \times m)(\Delta) \).
4. Find with proof the product measure \( (m \times c)(\Delta) \).

Here \( c \) is counting measure defined on all subsets of \( \mathbb{R} \) (so \( c(A) \) is the number of elements of a set \( A \)).