\( \mathbb{C} \) is the field of complex numbers \( z = x + iy \), \( \mathbb{R} \) the field of real numbers, \( \mathbb{D} \) the open unit disk, and \( \hat{\mathbb{C}} \) the Riemann sphere (aka, extended complex plane)

If you use a known theorem, or a similar result from a textbook, be sure to explicitly indicate so. Any other facts used must be proven. Proofs, or counter examples, are required for all problems.

1. Find the Laurent series of the function \( \frac{1}{z(z-1)} \) for the region \( 2 < |z+2| < 3 \).

2. Let \( f \) and \( g \) be non-constant and holomorphic in \( \mathbb{D} \setminus \{0\} \). Define \( h(z) \) for \( z \in \mathbb{D} \setminus \{0\} \) by \( h(z) := f(z)g(z) \).
   (a) Explain why \( h \) has an isolated singularity at \( z = 0 \).
   (b) Discuss the nature of the isolated singularity \( z = 0 \) for \( h \). (When is it: removable? a pole? an essential singularity?)

3. Let \( T(z) = \frac{z}{z+1} \).
   (a) Find the image \( T(\mathbb{R}) \) where \( \mathbb{R} \) is the extended real line in \( \hat{\mathbb{C}} \).
   (b) Find the image \( T(K) \) where \( K \) is the unit circle \( |z| = 1 \).
   (c) Find the image \( T(L) \) where \( L \) is the line \( \text{Re}(z) = 1 \).

4. Let \( f \) be holomorphic in the annulus \( A := \{1 < |z| < 2\} \). Suppose there is a sequence \( (p_n) \) of polynomials that converges locally uniformly in \( A \) to \( f \). Prove that there is a function \( F \) that is holomorphic in \( |z| < 2 \) with \( F|A = f \).

5. Compute the integral \( \int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x^4} \, dx \).

6. How many roots does the polynomial \( f(z) = z^5 + 2z^2 + 1 \) have in the annulus \( \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\} \)?