UNIVERSITY OF Cincinnati

MATHEMATICS QUALIFYING EXAM

MAY, 7, 2019

Four Hour Time Limit

Notation: $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space.

Unless explicitly stated, proofs, or counterexamples, are required for all problems.

1. Prove that for a subset $E \subset [0,1]$ the following conditions are equivalent:
(i) Every continuous function $f\colon E \to [0,\infty)$ is bounded.
(ii) E is a closed set.

2. Let (f_n) be a sequence of continuous functions on $D \subset \mathbb{R}^p$ to $\mathbb{R}^q$ such that (f_n) converges uniformly to f on D, and let (a_n) be a sequence of points in D that converges to $a \in D$. Prove that $(f_n(a_n))$ converges to $f(a)$.

3. Let f be a differentiable function on the interval $(-2,2)$ such that f' is continuous on this interval. Prove that

$$\lim_{h\to 0}\int_0^1 \left(\frac{f(x+h)-f(x)}{h} - f'(x)\right) dx = 0.$$

4. Find the largest set $D \subset \mathbb{R}$ such that for all $x \in D$ the series $\sum_{n=2}^{\infty} \frac{2^n}{n-1}(3x-1)^n$ converges.

5. Let P_3 be the collection of all polynomials in x with coefficients in $\mathbb{R}$ with degree at most 3, and let $T : P_3 \to P_3$ be the linear transformation given by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1x + 2a_2x^2 + 3a_3x^3.$$

(a) Find a basis for P_3 with respect to which the matrix representing T is diagonal.
(b) Determine the kernel and the image of T.

6. (a) Define what it means to say that vectors $v_1,\dots,v_k \in \mathbb{R}^n$ are linearly independent.
(b) Let A be an $n\times n$ matrix with real entries. If $v_1,\dots,v_k \in \mathbb{R}^n$ are eigenvectors of A with distinct real eigenvalues, use the definition to show that $v_1,\dots,v_k$ are linearly independent.
Hint: Use mathematical induction.

7. Let W be a subspace of an inner product space $(V,\langle\ ,\rangle)$. If W is spanned by vectors $\{v_1,\dots,v_k\}$, show that the orthogonal complement $W^\perp$ is equal to $\bigcap_{j=1}^{k}\{v_j\}^\perp$.

8. Let f be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ that sends the point (x,y) into the point (u,v) given by

$$u = x^2 - y^2, v = 3xy$$

Show that f is locally one-to-one at every point except $(0,0)$, but f is not one-to-one on $\mathbb{R}^2$.