1. Prove that for a subset $E \subset [0, 1]$ the following conditions are equivalent:
   (i) Every continuous function $f : E \to [0, \infty)$ is bounded.
   (ii) $E$ is a closed set.

2. Let $(f_n)$ be a sequence of continuous functions on $D \subset \mathbb{R}^p$ to $\mathbb{R}^q$ such that $(f_n)$ converges uniformly to $f$ on $D$, and let $(a_n)$ be a sequence of points in $D$ that converges to $a \in D$. Prove that $(f_n(a_n))$ converges to $f(a)$.

3. Let $f$ be a differentiable function on the interval $(-2, 2)$ such that $f'$ is continuous on this interval. Prove that
   $$\lim_{h \to 0} \int_0^1 \left( \frac{f(x + h) - f(x)}{h} - f'(x) \right) \, dx = 0.$$ 

4. Find the largest set $D \subset \mathbb{R}$ such that for all $x \in D$ the series $\sum_{n=2}^{\infty} \frac{2^n}{n-1} (3x - 1)^n$ converges.

5. Let $P_3$ be the collection of all polynomials in $x$ with coefficients in $\mathbb{R}$ with degree at most 3, and let $T : P_3 \to P_3$ be the linear transformation given by
   $$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1x + 2a_2x^2 + 3a_3x^3.$$ 
   (a) Find a basis for $P_3$ with respect to which the matrix representing $T$ is diagonal.
   (b) Determine the kernel and the image of $T$.

6. (a) Define what it means to say that vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly independent.
   (b) Let $A$ be an $n \times n$ matrix with real entries. If $v_1, \ldots, v_k \in \mathbb{R}^n$ are eigenvectors of $A$ with distinct real eigenvalues, use the definition to show that $v_1, \ldots, v_k$ are linearly independent.
   
   
   Hint: Use mathematical induction.

7. Let $W$ be a subspace of an inner product space $(V, \langle ., . \rangle)$. If $W$ is spanned by vectors $\{v_1, \ldots, v_k\}$, show that the orthogonal complement $W^\perp$ is equal to $\bigcap_{j=1}^{k} \langle v_j \rangle^\perp$.

8. Let $f$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ that sends the point $(x, y)$ into the point $(u, v)$ given by
   $$u = x^2 - y^2, \quad v = 3xy$$ 
   Show that $f$ is locally one-to-one at every point except $(0, 0)$, but $f$ is not one-to-one on $\mathbb{R}^2$. 