REAL & COMPLEX ANALYSIS PRELIMINARY EXAMINATION, AUGUST 2013, MATHEMATICS, UNIVERSITY OF CINCINNATI

Complex Analysis

1. Use the contour $[-R, R] + [R, R+\pi i] + [R+\pi i, -R+\pi i] + [-R+\pi i, -R]$ (here $R \in \mathbb{R}$) to evaluate
$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}} dx.$$

2. Suppose f is an entire function. Prove that $\sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}$ converges locally uniformly in $\mathbb{C}$ where $f^{(n)}$ denotes the n^{th} derivative of f.

3. Let $f(z) = e^{2z}$. Find all connected sets containing $z = i$ on which f is one-to-one.

4. Find the Laurent series for the following functions in the indicated region. Justify your answers.
(a) $f(z) = \frac{1}{(z-1)(z-2)}$, $1 < |z| < 2$.
(b) $g(z) = \frac{1}{(z-1)^2} - \frac{1}{(z-2)^2}$, $1 < |z| < 2$.

Real Analysis

1. (a) Let $(X, \mathcal{M}, \mu)$ be a *finite* measure space. Without using the Cauchy–Schwarz, Hölder, or Jensen inequalities, prove that if f^2 is integrable on $(X, \mathcal{M}, \mu)$, then so is f.

Is the same necessarily true without the condition $\mu(X) < \infty$? Prove or give a counterexample.

(b) Under the assumptions of part (a), define the following two finite measures on $(X, \mathcal{M})$:
$$\nu_1(E) = \int_E |f| \, d\mu, \ E \in \mathcal{M}; \quad \nu_2(E) = \int_E |f|^2 \, d\mu, \ E \in \mathcal{M}.$$
Is it true or false that $\nu_1 \ll \nu_2$? Justify. Is it true or false that $\nu_2 \ll \nu_1$? Justify.

2. Let $\{f_n\}$ be a sequence of real valued measurable functions on $\mathbb{R}$ such that $f_1 \geqq f_2 \geqq \cdots \geqq f_n \geqq \cdots \geqq 0$. Let $f(x) = \inf\{f_n(x) | n \in \mathbb{N}\}$, $x \in \mathbb{R}$.
a) Show that if f_1 is integrable, then $\int f_n \to \int f$.
b) Show that if f_1 is not integrable, then the conclusion in a) may no longer hold.

3. (a) Define the total variation of a function $f : [0, 1] \to \mathbb{R}$ and the absolute continuity of f.
(b) Suppose $f : [0, 1] \to \mathbb{R}$ and is absolutely continuous and define g by
$$g(x) = \int_0^1 f(xy) dy.$$
Show that g is absolutely continuous.

4. Suppose that f is a Lebesgue integrable, decreasing function on $(0, \infty)$. Prove that
$$\lim_{x \to \infty} x \, f(x) = 0.$$