1
A right circular cone is inscribed in a sphere of radius $R$ as in Figure 1. Find the maximal possible cone volume. What is the ratio between the sphere volume and the maximal inscribed cone volume?

Figure 1.

1.1 solution
Let $\alpha R$, for $0 \leq \alpha \leq 1$, be the distance the origin and the cone-base (so that the height of the cone is equal to $R + \alpha R$). Setting $z = \alpha R$, $x = r \cos(\theta)$, $y = r \sin(\theta)$ in $x^2 + y^2 + z^2 = R^2$, we get

$$r = R \sqrt{1 - \alpha^2}$$

so that the volume of the cone, as a function of $\alpha$, is equal to

$$V(\alpha) = \frac{(R + \alpha R)^2 \pi}{3} = \frac{1}{3} \pi R^3 (1 + \alpha) (1 - \alpha^2)$$

Solving equation $V'(\alpha) = 0$, we get

$$\alpha_1 = -1, \quad \alpha_2 = \frac{1}{3}$$

Since $0 \leq \alpha \leq 1$, we get $\alpha = \alpha_2 = 1/3$, and the corresponding cone-volume is equal to
Finally since, as it can be shown, the volume of the ball is equal to $4\pi R^3/3$, the ratio between the two is equal to

$$\frac{(4\pi R^3/3)}{V\left(\frac{1}{3}\right)} = \frac{27}{8} \approx 3.375$$

2

One corner of a page of width $a = 8$ inches is folded over to just reach the opposite side as indicated in Figure 2. After expressing the length $L$ of the crease in terms the angle $\theta$, find the width $x$ of the part folded over when $L$ is a minimum.

![Figure 2](image-url)
2.1 solution

Just above the lower arrow head, the three angles reading from right to left are $\pi/2 - \theta$, $\pi/2 - \theta$, and $2\theta$. Thus, we have

$$\cos(2\theta) = \frac{a - x}{x}, \quad 2\cos(\theta)^2 = 1 + \cos(2\theta) = \frac{a}{x}, \quad x = \frac{a}{2\cos(\theta)^2} \quad (2.1)$$

With $L = \frac{x}{\sin(\theta)}$, we obtain $L(\theta) = \frac{a}{2\sin(\theta)\cos(\theta)^2}$. Differentiation $L(\theta)$, we get

$$L'(\theta) = \frac{a}{2\sin(\theta)^2 \cos(\theta)^3} \quad (3.3)$$

and therefore $\sin(\theta)^2 = 1/3$, and

$$x = \frac{a}{2\cos(\theta)^2} = \frac{a}{2\left(1 - \sin(\theta)^2\right)} = \frac{a}{2 \times \frac{2}{3}} = \frac{3a}{4}$$

So, if $a = 8$, then $x = 6$.

3

Suppose that $a_1$, $a_2$, ..., $a_n$ are real numbers such that the function

$$f(x) = a_1 \sin(x) + a_2 \sin(2x) + ... + a_n \sin(nx) \quad (3.1)$$

satisfies $|f(x)| \leq |\sin(x)|$, for all real numbers $x$. Prove that

$$|a_1 + 2a_2 + ... + na_n| \leq 1 \quad (3.2)$$

3.1 solution

Note that

$$|a_1 + 2a_2 + ... + na_n| = \left| f'(0) \right| = \left| \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \right| = \left| \lim_{x \to 0} \frac{f(x)}{x} \right| = \left| \lim_{x \to 0} \frac{f(x)}{x} \right| \leq \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad (3.3)$$

4

For what values of $p$ does the series $\sum_{n=1}^{\infty} \left( e^{-\frac{1}{n^2}} + \frac{1}{n^2} - 1 \right)^p$ converge? Fully justify your answer.

4.1 solution
For small $x$ we have $e^x \approx 1 + x + x^2/2$. Therefore, intuitively, for large $n$, $e^{-1 \over n^2} \approx 1 - {1 \over n^2} + {1 \over 2n^4}$, meaning that $e^{-1 \over n^2} + {1 \over n^2} - 1 \approx {1 \over 2n^4}$. Formally, we apply the Limit Comparison Test:

$$\lim_{n \to \infty} \frac{\left(e^{-1 \over n^2} + {1 \over n^2} - 1\right)^p}{n^{-p}} = 2^{-p}$$

(4.1)

so that $\sum_{n=6}^{\infty} \left(e^{-1 \over n^2} + {1 \over n^2} - 1\right)^p$ converges iff $\sum_{n=6}^{\infty} \frac{1}{n^p}$ does, which happens iff $4p > 1$, i.e., iff $p > {1 \over 4}$.

5

Show that the limit $\gamma := \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right)$ exists, and find an upper and lower bound for $\gamma$.

5.1 solution

Set $\Gamma(n) := \sum_{k=1}^{n} \frac{1}{k} - \log(n)$, where $n$ is a positive integer. Then

$$\Gamma(n+1) - \Gamma(n) = \frac{1}{n+1} + \log(n+1) - \log(n) = \frac{1}{n+1} + \int_n^{n+1} \frac{1}{x} \, dx < 0$$

(5.1)

so that the sequence $\Gamma(n)$ is decreasing. Also,

$$\log(n+1) - \log(2) = \int_2^{n+1} \frac{1}{x} \, dx \leq \sum_{k=2}^{n} \frac{1}{k}$$

(5.2)

and therefore

$$\log\left(\frac{n+1}{n}\right) - \log(2) + 1 \leq \sum_{k=1}^{n} \frac{1}{k} - \log(n)$$

(5.3)

Therefore, since $\lim_{n \to \infty} \log\left(\frac{n+1}{n}\right) = 0$,

$$1 - \log(2) \leq \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right) = \gamma \leq \Gamma(n) = \sum_{k=1}^{n} \frac{1}{k} - \log(n)$$

(5.4)

for any integer $n \geq 1$. For example, setting $n = 2$,

$$1 - \log(2) \leq \gamma \leq 1 - \log(2) + {1 \over 2}$$

(5.5)
or \(0.306853 \leq \gamma \leq 0.806853\) (the “exact” value being equal to \(\gamma = 0.577216\ldots\)).

6
Calculate the following integral and check your result by differentiation.

\[
\int (\ln x)^2 \, dx
\]  

(6.1)

6.1 solution
Setting \(u = (\ln x)^2\), \(dv = dx\); \(du = 2 (\ln x) \frac{1}{x} \, dx\), \(v = x\), by integration by parts we get

\[
\int (\ln x)^2 \, dx = x (\ln x)^2 - 2 \int \ln x \, dx
\]

Setting \(u = \ln x\), \(dv = dx\); \(du = \frac{1}{x} \, dx\), \(v = x\), by integration by parts we get

\[
\int (\ln x)^2 \, dx = x (\ln x)^2 - 2 \int \ln x \, dx = 2x + x (\ln x)^2 - 2x \ln x.
\]

7
Calculate the following integral and check your result by differentiation.

\[
\int \frac{1}{x^2 - x} \, dx
\]  

(7.1)

7.1 solution
Trying

\[
\frac{1}{x^2 - x} = \frac{1}{x(x^6 - 1)} = \frac{A}{x} + \frac{Bx^5}{x^6 - 1}
\]

we get \(A = -1\), \(B = 1\), so that (assuming \(0 < x < 1\))

\[
\int \frac{1}{x^2 - x} \, dx = -\int \frac{1}{x} \, dx - \int \frac{x^5}{1 - x^6} \, dx = \frac{-1}{6} \ln(1 - x^6) - \ln(x)
\]

Checking,

\[
\left(\frac{-1}{6} \ln(1 - x^6) - \ln(x)\right)' = -\frac{x^5}{1 - x^6} - \frac{1}{x} = \frac{1}{x^2 - x}.
\]