MATHEMATICS QUALIFYING EXAM

AUGUST 20, 2018

Four Hour Time Limit

Notation: $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space.

Unless explicitly stated, proofs, or counterexamples, are required for all problems.

1. For continuous function $f : [0,1] \to [0,1]$, must there exist c in that interval for which $f(c) = c$?

2. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = x$ for x rational and $f(x) = -x$ for x irrational. Prove that f is not Riemann integrable on $[0,1]$.

3. Suppose that $\{c_n\}$ is a sequence of real numbers which converges to $c \in \mathbb{R}$. For $n \in \mathbb{N}$, let

$$a_n = \frac{c_1 + c_2 + \cdots + c_n}{n}.$$

Prove that $\{a_n\}$ converges to c. You can use without proof the fact that $\{c_n\}$ is bounded.

4. Let $f_n(x) = \frac{x}{1+nx^2}$ and $f(x) = 0$ for $x \in \mathbb{R}$.

(a) For what values of x is it true that $f_n'(x) \to f'(x)$?

(b) Show that f_n converges uniformly to f on $\mathbb{R}$.

5. Let V be the real vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Consider three functions f_1, f_2, f_3 in V, defined for real x by $f_1(x) = \sin(x - \pi/4)$, $f_2(x) = \sin x$, $f_3(x) = \sin(x + \pi/4)$. Is the set $\{f_1, f_2, f_3\}$ linearly independent?

6. If A is an n by n real symmetric matrix, show that all eigenvalues of A^2 are non-negative.

7. Prove the following statement, called the Hamilton–Cayley theorem for 2×2 matrices.

Theorem 1. *If A is a 2×2 matrix and $p(\lambda)$ is its characteristic polynomial, then $p(A) = 0$.*

8. Consider a subset $\mathcal{D} = (-2,2)^2 \setminus [-1,1]^2$ of $\mathbb{R}^2$. Suppose $f : \mathcal{D} \to \mathbb{R}$ is differentiable with derivative $Df(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{D}$. Show that $f(\mathbf{x}) = f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}$.