QUALIFYING EXAMINATION, AUGUST 2017

In this exam $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{R}^n$ is $n$-dimensional Euclidean space. This exam has eight questions. Proofs, or counter examples, are required for all problems.

1. Let $f, g$ and $h$ be three functions on the interval $[-1, 1]$, and suppose that $f \leq h \leq g$ on this interval. Show that if $f, g$ are differentiable at 0 with $f(0) = g(0)$ and $f'(0) = g'(0)$, then $h$ is differentiable at zero with $h'(0) = f'(0)$.

2. Given a non-empty set $A \subset \mathbb{R}$, let $\rho_A : \mathbb{R} \to \mathbb{R}$ be given by $\rho_A(x) = \inf \{|x - z| : z \in A\}$.
   (a) Show that for all $x, y \in \mathbb{R}$ we have $|\rho_A(x) - \rho_A(y)| \leq |x - y|$.
   (b) Using (a) above and the definition of absolute continuity, show that $\rho_A$ is absolutely continuous on $\mathbb{R}$.
   (c) Is $\rho_A$ differentiable everywhere? Prove or give a counterexample (with a proof for counterexample).

3. Show that there are at least two distinct real numbers $x$ for which
   \[x^{10} + \frac{20}{1 + x^2 + \cos^2(x)} = 21.\]
   
   Hint: You might want to consider the graph of $f(x) = x^{10} + \frac{20}{1 + x^2 + \cos^2(x)} - 21$.

4. Find all the positive integer values of $n$ for which the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by
   \[f(x, y) = \begin{cases} 
   x^n y^n & \text{if } (x, y) \neq (0, 0), \\
   0 & \text{if } (x, y) = (0, 0) 
   \end{cases}\]
   is differentiable at $(0, 0)$.

5. Let $f : [0, 1] \to \mathbb{R}$ be continuous, and let $\Gamma = \{(x, f(x)) : x \in [0, 1]\}$. State the definition of connectedness, and prove that $\Gamma$ is a connected subset of $\mathbb{R}^2$. You may use without proof the fact that the interval $[0, 1]$ is a connected set.

6. In this question $\ell^1$ is the collection of all sequences $x = \{x_k\}_{k \in \mathbb{N}}$ of real numbers such that $\sum_{k=1}^{\infty} |x_k|$ is finite. Show that $\ell^1$ is a vector space over the field $\mathbb{R}$, and that it is an infinite-dimensional vector space.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that for all vectors $x \in \mathbb{R}^n$ we have $\|T(x)\| = \|x\|$. Here, $\| \cdot \|$ is the standard Euclidean norm on $\mathbb{R}^n$.
   (a) Show that if $x, y \in \mathbb{R}^n$ with their inner product $x \cdot y = 0$, then $T(x) \cdot T(y) = 0$.
   (b) Show that the columns of the matrix representing $T$ in the standard basis of $\mathbb{R}^n$ form a mutually orthogonal collection of vectors.

Date: The exam continues on the next page. May 16, 2017.
8. For each positive integer \( n \), denote by \( \mathcal{P}_n \) the collection of all polynomials in \( x \) with coefficients in \( \mathbb{R} \). You may take for granted that for each \( \mathcal{P}_n \) is an \( n + 1 \)-dimensional vector space. Let \( T: \mathcal{P}_3 \to \mathcal{P}_2 \) be the linear map given by
\[
Tp(x) = \frac{p(x) - p(0)}{x}.
\]
(a) Show that \( T \) is surjective but not injective. You do not have to prove that \( T \) is a linear map.
(b) Show that the map \( S: \mathcal{P}_2 \to \mathcal{P}_3 \) given by \( Sp(x) = xp(x) \) is a right inverse of \( T \) (that is, \( T \circ S \) is the identity map on \( \mathcal{P}_2 \)) but that \( S \) is not the inverse of \( T \).