Statistics Part

1. Suppose that \(\{X_1, ..., X_n\}\) is a random sample from an exponential distribution with an unknown parameter \(\beta > 0\) and pdf given by
\[
f(x|\beta) = \frac{1}{\beta} \exp\{-x/\beta\}; \ x > 0,
\]
and that \(\{Y_1, ..., Y_n\}\) is random sample from an exponential distribution with an unknown parameter \(\delta > 0\). Assume that the two samples are independent. Let \(\theta = P(X_1 < Y_1)\). Find the uniformly minimum variance unbiased estimator of \(\theta\) based on the random samples given above when \(n = 2\).

2. Suppose \(X_1, ..., X_n\) are iid from a distribution with pdf
\[
f(x|\theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x > \theta \\ 0 & \text{else,} \end{cases}
\]
where \(\theta > 0\) is unknown. We want to test the null hypothesis \(H_0 : \theta \leq \theta_0\) versus the alternative hypothesis \(H_1 : \theta > \theta_0\) for a known value \(\theta_0 > 0\). Is there a uniformly most powerful (UMP) test? If there is one, find the UMP test and give the critical region of size \(\alpha = 0.05\). If there is none, explain why.

3. Let \(X_1, ..., X_n\) be iid Bernoulli(p), \(0 < p < 1\). Let \(T_n\) be the uniformly minimum variance unbiased estimator of \(p^2\). Find \(T_n\) and determine if it is asymptotically normally distributed in the sense that \(\sqrt{n}(T_n - \mu)\) converges in distribution to a normal distribution, for some constant \(\mu\).

4. Let \(x_1, ..., x_n\) be a random sample from Bernoulli(\(\theta\)) distribution, where \(0 < \theta < 1\), and is unknown. Assume a \(U(0, 1)\) prior for \(\theta\). Consider the loss function for estimating \(\theta\) given by
\[
L(\theta, a) = \frac{(\theta - a)^2}{(1 - \theta)}.
\]
Find the Bayes rule with respect to the above loss function.
Probability Part

5. Let $X$ be a non-negative random variable. Prove that

$$\mathbb{E} e^X = 1 + \int_0^\infty e^t \mathbb{P}(X > t) dt.$$ 

6. Consider a sequence of independent random variables $\{X_n\}_{n \in \mathbb{N}}$, each $X_n$ has the following distribution:

$$\mathbb{P}(X_n = k) = \frac{1}{(n+5)k\gamma}, k = 1, 2, 3,$$

and $\mathbb{P}(X_n = 0) = 1 - \mathbb{P}(X_n \in \{1, 2, 3\})$, for some parameter $\gamma > 0$. So, each realization of $\{X_n\}_{n \in \mathbb{N}}$ is an infinite sequence of 0, 1, 2, 3. Find the range of $\gamma$ so that both of the following conditions are satisfied at the same time.

(a) The number 1 occurs infinitely often with probability one.

(b) The number 3 occurs only a finite number of times with probability one.

Justify your answer.

7. Suppose that $X_1, X_2, \ldots$ are independent random variables, and $X_n$ is uniformly distributed over $[0, n]$ (i.e., $\mathbb{P}(X_n \leq x) = x/n, x \in [0, n], n \in \mathbb{N}$). Find appropriate sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ so that

$$\frac{\sum_{k=1}^n X_k - b_n}{a_n} \Rightarrow \mathcal{N}(0, 1),$$

as $n \to \infty$, where $\mathcal{N}(0, 1)$ is the standard normal distribution. Justify your answer. Hints: You may use the following estimates

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$ 

8. Suppose that for each $n \in \mathbb{N}$, random variable $X_n$ has density $f_n(x) = 1 + \cos(2\pi nx)$ on $[0, 1]$. Prove that $X_n$ converges in distribution to some random variable $X$ as $n \to \infty$, and determine the law of $X$. 

2