Sample Questions for the PhD Preliminary Exam in Analysis

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Real Analysis

1. Let $f$ be a real-valued function on $\mathbb{R}$. Show that the set of points where $f$ is continuous is a $G_\delta$ set.

2. Let $f$ be a non-negative integrable function on a measure space $(X, \mathcal{M}, \mu)$. Suppose that, for every $n \in \mathbb{N}$,

$$\int_X |f|^n \, d\mu = \int_X f \, d\mu.$$ 

Show that $f = \chi_E$ a.e. on $X$, where $E$ is a measurable subset of $X$.

3. Let $f$ be absolutely continuous and strictly increasing on $[a, b]$. Show that for every open subset $O$ of $(a, b)$,

$$m(f(O)) = \int_O f'.$$

4. Let $f$ be a real-valued function that is integrable on $\mathbb{R}$ and let $\varepsilon > 0$. Show that there is a continuous function $g$ that is identically zero outside some interval and such that

$$\int_{\mathbb{R}} |f - g| < \varepsilon.$$

(Hint: Lusin’s theorem.)

5. Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space and $f$ a measurable real-valued function on $X$. Prove that

$$\int_X f^2 \, d\mu = 2 \int_0^\infty s \mu(\{x \in X : |f(x)| > s\}) \, ds$$
Complex Analysis

1. Let $\text{Log}(z)$ denote the principal branch of the logarithm function.
   
   (a) Show that in general $\text{Log}(ab) \neq \text{Log}(a) + \text{Log}(b)$.
   
   (b) For a given $a \in \mathbb{C}\setminus\{0\}$, determine the set of all $z$ for which
       
       $\text{Log}(az) = \text{Log}(a) + \text{Log}(z)$.

2. State and derive the Cauchy-Riemann equations.

3. Evaluate
   
   $\int_{|z|=1} (z^2 + 2z) \csc(z)^2 dz$.

4. Let $f(z) = z^2$.
   
   (a) Calculate $\int_0^{2\pi} f(2 + e^{i\theta}) d\theta$ and confirm it is non-zero.
   
   (b) Does Cauchy’s theorem give $\int_{|z|=1} f(z) dz = 0$?
       Explain the seeming discrepancy with (a).

5. Let $f(z) = u(z) + iv(z)$ be an entire function satisfying $u(z) \leq 0$ for all $z \in \mathbb{C}$.
       Show that $f(z)$ is constant.
       **Hint:** Consider $g(z) = e^{f(z)}$.

6. (a) Find the general form of any Möbius transformation which maps the unit disk $D$ onto the upper half-plane.
      
      (b) What changes if we add the requirement that the origin should be mapped to $i$?

7. Show that $f(z) = z^4 - 3z^2 + 3$ has exactly one zero in the open first quadrant
       $Q_1 = \{z : \Re(z) > 0, \Im(z) > 0\}$.
       **Hint:** Use the Argument Principle.

8. Let $R$ be a rational function. State and prove necessary and sufficient conditions for there to be a holomorphic branch of the logarithm of $R$ in some domain $\Omega$. 