Sample Questions for the PhD Preliminary Exam in Algebra and Topology

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Algebra

(1) Consider the polynomial $f(x) = x^6 - 4x^3 + 1 \in \mathbb{Z}[x]$ which you may assume without proof to be irreducible. Let K be the splitting field of F over $\mathbb{Q}$.
a) Find all the complex roots of f. Show, in particular that f has two real roots whose product is 1.
b) Let α be a real root of f. Show that $K = \mathbb{Q}(\alpha, \omega)$ where ω is a primitive cube root of one. Deduce that $|\operatorname{Gal}(K, \mathbb{Q})| = 12$.
c) Show that $\operatorname{Gal}(K, \mathbb{Q})$ is a dihedral group.

(2) Let K be a field with 64 elements and denote by $\mathbb{F}_2$ the field with 2 elements.
a) Find all subfields of K.
b) How many elements $\alpha \in K$ are there such that $\mathbb{F}_2(\alpha) = K$?
c) Determine using (b) the number of irreducible polynomials of degree 6 over $\mathbb{F}_2$.

(3) Let F be a field.
a) Outline the proof of the fact that $F[x]$ is a PID.
b) Let $R = \{f(x) \in F[x] \mid f'(0) = 0\}$. Show that R is not a UFD and find an ideal that is not principal.

(4) Let k be a field of characteristic $p > 0$ and let $0 \neq c \in k$. Show that the polynomial $x^p - x - c$ is irreducible if and only if it has no roots in k. Show that this is false if the characteristic of k is 0.

(5) A field extension $K \supset F$ is called *biquadratic* if $[K : F] = 4$ and K is generated over F by the roots of two irreducible quadratic polynomials. Prove that the extension $K \supset F$ is biquadratic if and only if it is Galois with Galois group the Klein four group.

(6) Let R be a principal ideal domain with a unique non-zero prime ideal (p).
(a) Show that every element of R can be expressed uniquely in the form up^n for some non-negative integer n and unit u.
(b) Let $\nu : R \to \mathbb{Z}^+$ be the function given by $\nu(up^n) = n$. Show that ν satisfies
(i) $\nu(ab) = \nu(a) + \nu(b)$;
(ii) $\nu(a + b) \geq \min(\nu(a), \nu(b))$;
(c) Conversely, show that if F is a field and $\nu : R \to \mathbb{Z}^+$ is a surjective map satisfying the properties above then the set

$$D = \{a \in F \mid \nu(a) \geq 0\}$$

is a principal ideal domain with a unique non-zero prime ideal.

(7) (a) State and prove Eisenstein's criterion for the irreducibility of polynomials over $\mathbb{Z}$.
(b) Use this result to prove that the polynomial $[(x+1)^p - 1]/x$ is irreducible if p is prime.

(c) Deduce that the cyclotomic polynomial $\Phi_p(x) = 1+x+x^2+\cdots+x^{p-1}$ is irreducible if p is prime.

(8) (a) Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$.
(b) Show that its roots are $\pm\sqrt{1 \pm \sqrt{3}}$.
(c) Let $K_1 = \mathbb{Q}(\sqrt{1+\sqrt{3}})$, $K_2 = \mathbb{Q}(\sqrt{1-\sqrt{3}})$. Show that $K_1 \neq K_2$ and that $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3})$.
(d) Determine the galois group of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$.

(9) Let k be a field and let $f(x, y) \in k[x, y]$. Prove that if $f(x, x) = 0$, then $f(x, y)$ is divisible by $x - y$. (Hint: use induction on the degree of f as a polynomial in x with coefficients in $k[y]$).

(10) Let $f(x) = x^4 + 5x + 5$.
(a) Find the roots of f. What is the Galois group of f over the real numbers $\mathbb{R}$?
(b) Show that f is irreducible over $\mathbb{Q}$.
(c) Show that the splitting field of f has degree 4 over $\mathbb{Q}$ and find the Galois group of f over $\mathbb{Q}$.

Topology

(1) Prove or disprove.
   (a) The product of two quotient maps is a quotient map.
   (b) The product of connected spaces is connected.

(2) Prove that a product space $\Pi_{\lambda \in \Lambda} X_\lambda$ is contractible if and only if for each $\lambda \in \Lambda$, the space X_λ is contractible.

(3) Given a topological space X, the cone $C(X)$ of the space X is the topological space $X \times [0,1]/X \times \{0\}$ (i.e. $C(X)$ is the quotient space obtained from $X \times [0,1]$ by collapsing $X \times \{0\}$ to a point), and the suspension $\sum(X)$ of X is the topological space $X \times [0,1]/\sim$, where for $(a,s),(b,t) \in X \times [0,1]$, $(a,s) \sim (b,t)$ if $s = t$ and either $a = b$, or $t = 0$, or $t = 1$ (i.e. $\sum(X)$ is the quotient of $X \times I$ obtained by identifying $X \times \{0\}$ to a single point and $X \times \{1\}$ to another single point).
   (a) Show that $C(X)$ is contractible (thus simply connected).
   (b) Is $\sum(X)$ always simply connected? Prove or disprove.

(4) Let X be the complement of two circles $\{x^2 + y^2 = 1;\ z = 1\}$ and $\{x^2 + y^2 = 1;\ z = -1\}$ in $\mathbb{R}^3$. Show that X is path connected and determine the fundamental group $\pi_1(X)$.

(5) Show that there is no one-to-one continuous map from $\mathbb{R}^n \to \mathbb{R}^2$ for $n > 2$.

(6) Let C be the "boundary circle" of the (compact) Möbius band $\mathbb{MB}$. Attach $\mathbb{MB}$ to the "top" of the cylinder $\mathbb{S}^1 \times \mathbb{I}$ using any homeomorphism $\mathbb{MB} \supset C \to \mathbb{S}^1 \times \mathbb{I}$. Then attach the torus $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ to the "bottom" of the cylinder using any homeomorphism $\mathbb{T}^2 \supset \mathbb{S}^1 \times \{(1,0)\} \to \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times \mathbb{I}$. Let X be the resulting space. Thus X is obtained by first attaching a Möbius band to the top of a cylinder and then attaching a torus to the bottom of the cylinder. Calculate the fundamental group of X.

(7) For each integer $m > 2$ and each $n \in \mathbb{N}$, construct a compact connected m-manifold whose fundamental group is the free group on n generators. Can you do this if $m = 2$?

(8) (a) Find the universal covering space of the one point union $X := \mathbb{K} \vee \mathbb{S}^1$ of the Klein bottle and the cycle.
   (b) Find a covering space $Y \xrightarrow{p} X$ that corresponds to an infinite cyclic subgroup of the fundamental group of X.
   (A description of a covering space includes both a definition of the total space as well as a definition of the covering map, and an indication of why the map is a covering projection.)