Name: 

Student Id #: 

Instruction: Do all eight problems.

Score:

Problem 1.1 ————- Problem 2.1———
Problem 1.2 ————- Problem 2.2———
Problem 1.3———- Problem 2.3———
Problem 1.4———- Problem 2.4———

Part I total score : ———— Part II total score ————

Total score ————
Problem 1.1

Identify the stable and unstable subspaces for the linear ODE

\[ x'(t) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} x(t). \]
Problem 1.2

Write the following system in polar coordinates and determine if the origin is a center, a stable focus or an unstable focus.

\[
\dot{x} = -y + xy^2, \quad \dot{y} = x + y^5.
\]
Problem 1.3

Show that if a function $x(t)$ satisfies $0 \leq \frac{dx}{dt} \leq x^2$ for all $t$, and $x(0) = -1$, then $x(t) < 0$ for all $t \in [0, \infty)$. 
Problem 1.4

Show that

\[ \dot{x} = y, \quad \dot{y} = -x + (1 - x^2 - y^2)y \]

has a unique stable limit cycle which is the \( \omega \)-limit set of every trajectory except the critical point at the origin. (Hint: compute \( \dot{r} \)).
Problem 2.1.

Find the solution to the differential equation
\[
\begin{cases}
  u_x u_y = 1 & \text{in } (0, \infty) \times (0, \infty), \\
  u(x, 0) = 2\sqrt{x} & \text{for all } x > 0.
\end{cases}
\]
Problem 2.2.

(i) Solve the following initial-value-problem.

\[
\begin{aligned}
&u_t + 2u_x - 4u_y + 5u_z = 0, \quad (x, y, z, t) \in \mathbb{R}^3 \times (0, \infty), \\
&u(x, y, z, 0) = x^2 + y - 3z \quad (x, y, z, t) \in \mathbb{R}^3.
\end{aligned}
\]

(ii) Use energy estimate method to show that the following initial-boundary-value problem admits at most one smooth solution \( u = u(x, t) \).

\[
\begin{aligned}
&u_t + 2u_x + 12u_{xxx} = 0, \quad x \in (0, L), \ t \in (0, T) \\
&u(x, 0) = g(x), \ x \in (0, L), \\
&u(0, t) = 0, \ u(L, t) = 0, \ u_x(L, t) = 0.
\end{aligned}
\]
Problem 2.3.

Let $U$ be the unit ball in $\mathbb{R}^3$. Suppose $u$ solves the heat equation

\[
\begin{cases}
    u_t - \Delta u = 0 \text{ in } U_T, \\
    u(x,t) = 6t \text{ when } |x| = 1, \\
    u(x,0) = g(x)
\end{cases}
\]

Suppose that $g \leq 0$. Prove that $u(x, t) \leq 6t + |x|^2$ for all $(x, t) \in U_T$. 

Problem 2.4.

Find a solution formula for the following initial boundary value problem of the wave equation.

\[
\begin{align*}
    &u_{tt} - u_{xx} = 0, \quad x, t \in (0, \infty), \\
    &u(x, 0) = g(x), \quad u_t(x, 0) = h(x), \quad x \in (0, \infty), \\
    &u_x(0, t) = 0, \quad t \geq 0.
\end{align*}
\]