PRELIMINARY EXAM IN DIFFERENTIAL EQUATIONS
AUGUST 2014

Time Limit: Four Hours.

Please write your 3-digit identification number on each page.

Proofs, or counter examples, are required for all problems.

1) Show that the problem (here \( y = y(t) \))

\[
y' = y^{2/3}, \quad y(0) = 0
\]

has infinitely many solutions. Explain why the existence and uniqueness theorem does not apply here.

2) Assume that the function \( u(x) \geq 0 \) is continuous for \( x \geq 1 \), and for some number \( K > 0 \), we have

\[
xu(x) \leq K + \int_{1}^{x} u(t) \, dt, \quad \text{for } x \geq 1.
\]

Show that \( u(x) \leq K \), for \( x \geq 1 \).

3) Let \( A \) be a real \( 3 \times 3 \) constant matrix. Suppose that all solutions of

\[
x' = Ax
\]

are bounded as \( t \to \infty \), and as \( t \to -\infty \). Show that every solution is periodic, and there is a common period for all solutions.

Hint: Represent the solutions using the eigenvalues and the eigenvectors of \( A \).

4) Show that \((0, 0)\) is the only rest point in \( \mathbb{R}^2 \) of the system

\[
x' = 2y - x^3
\]

\[
y' = -x - y^5,
\]

and prove that this rest point is asymptotically stable in Lyapunov’s sense.

(State Lyapunov’s theorem that you are using.)
5) Find the solution to the differential equation

\[
\begin{align*}
&x_1 u_{x_1} + 2x_2 u_{x_2} = 4u + x_1^3 &\text{in} & (0, \infty) \times (0, \infty), \\
u(a, a) = a^3 &\text{for all } a > 0.
\end{align*}
\]

6) Suppose \( u \) is a harmonic function in \( \mathbb{R}^2 \), and has the values

\[ u(0, 0) = 1 \quad \text{and} \quad u(1, 0) = 4. \]

Show that \( u \) is not strictly positive in the ball \( \{|x| < 2\} \subset \mathbb{R}^2 \).

7) Let \( U \subset \mathbb{R}^n \) be an bounded open set. Suppose \( u \) has the properties

\[
\begin{align*}
\Delta(\Delta u) &= 0 \quad \text{in} \ U, \\
\Delta u &= 0 \quad \text{on} \ \partial U.
\end{align*}
\]

Show that \( u \) does not have a local maximum at any point \( x \in U \), and therefore

\[ \max_{x \in U} u(x) = \max_{x \in \partial U} u(x). \]

8) Let \( U \subset \mathbb{R}^n \) be a bounded open set with smooth boundary. Suppose \( u \) is a smooth solution of the differential equation

\[
\begin{align*}
&u_t - \Delta u + u^3 = 0 \quad \text{in} \ U \times (0, \infty) \\
&\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \partial U \times (0, \infty) \\
u(x, 0) = g(x) \quad \text{on} \ U \times \{t = 0\}.
\end{align*}
\]

Show that \( \int_U |Du(x, t)|^2 \, dx \leq \int_U |Dg(x)|^2 \, dx \) for all \( t > 0 \).