Instruction: Do all eight problems.

Score:

Problem 1.1 ————- Problem 2.1————
Problem 1.2 ————- Problem 2.2————
Problem 1.3————- Problem 2.3————
Problem 1.4————- Problem 2.4————

Part I total score : ———- Part II total score —————

Total score ——————
Problem 1.1: Suppose $A$ is an $n \times n$ matrix and $t \in \mathbb{R}$.

1. Define the matrix exponential $e^{tA}$.

2. Show that $\frac{d}{dt} e^{tA} = Ae^{tA}$. 

Problem 1.2: Each of the following systems $\dot{x} = f(x)$ has an equilibrium at $(0,0)$. For each system, determine which of the following behaviors are possible:

- saddle
- unstable node
- stable node
- unstable focus
- stable focus
- center
- center focus

Explain your reasoning. Be sure to note any properties of $f(x)$ you use.

1. $\dot{x} = x - 3y + y^5$
   $\dot{y} = 4x - y + xy^4 + x^2y^2$

2. $\dot{x} = 5x + 4 \sin y + 2x^2y - xy^3$
   $\dot{y} = x + 2y + x^3 + 7xy$
Problem 1.3: State and prove Gronwall’s inequality.
Problem 1.4: Let $f$ be a $C^1$ vector field in an open set $E \subset \mathbb{R}^2$ containing an annular region $A$ with a smooth boundary. Suppose that $f$ has no zeros in the closure of $A$, and that $f$ is transverse to the boundary of $A$, pointing inward. Show that $A$ contains a periodic orbit. Also show that if $A$ contains a finite number of cycles $\{C_1, \ldots, C_m\}$, then $A$ contains at least one stable limit cycle.
Problem 2.1 Let $u$ be a smooth function satisfying

$$\Delta u = 0$$

and $u \geq 0$ on the upper half-plane $\{(x_1, x_2) : x_2 > 0\}$. Suppose $u(0, 2) = 1$. Prove that $u(0, 3) \leq 4$. 
Problem 2.2  Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and $u$ a solution of the heat equation with boundary conditions

\[
\begin{cases}
  u_t - \Delta u = 0 & \text{in } U \times (0, T] \\
  \frac{\partial u}{\partial \nu} = -u & \text{on } \partial U \times (0, \infty) \\
  u(x, 0) = g(x) & \text{on } U \times \{t = 0\}
\end{cases}
\]

Suppose that $g(0) > 0$. Show that $\max_{U \times T} u(x, t) = \max_U g(x)$. 

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Problem 2.3 Let $U \subset \mathbb{R}^n$ be open and bounded, with smooth boundary, and $c(x, t)$ a nonnegative function on $U \times (0, \infty)$. Show that a smooth solution to the PDE

$$
\begin{aligned}
&\begin{cases}
  u_{tt} + c(x, t)u_t - \Delta u = 0 & \text{in } U \times (0, T) \\
  u(x, t) = 0 & \text{on } \partial U \times [0, T] \\
  u(x, 0) = 0 & \text{on } U \times \{t = 0\} \\
  u_t(x, 0) = h(x) & \text{on } U \times \{t = 0\}
\end{cases}
\end{aligned}
$$

satisfies the inequality $\int_U u_t^2 + |Du|^2 \, dx \leq \int_U h^2 \, dx$ at every $t > 0$. 
Problem 2.4-a Find the entropy solution to the equation
\[
\begin{cases}
    u_t + e^u u_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
    u(x, 0) = \begin{cases} 1 \text{ if } x < 0 \\ 0 \text{ if } x \geq 0 \end{cases}
\end{cases}
\]

Problem 2.4-b. Find the entropy solution to the equation
\[
\begin{cases}
    u_t + e^u u_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
    u(x, 0) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}
\end{cases}
\]