REAL & COMPLEX ANALYSIS PRELIMINARY EXAMINATION
AUGUST 2016, MATHEMATICS, UNIVERSITY OF CINCINNATI

Part 1. Real analysis

(1) Prove that for each $0 < \varepsilon < 1$, there exists a closed set $A \subset [0,1]$ with empty interior but Lebesgue measure greater than ε.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function.
Show that if $\int_a^b f(x)dx = 0$ for all rational numbers $a < b$, then $f(x) = 0$ a.e.
Hint: First prove $\int_A f = 0$ for A an open set, then for A measurable.

(3) Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $f\colon X \to \mathbb{R}$ is measurable.
  (a) Show $(f_*\mu)(A) := \mu(f^{-1}(A))$ defines a measure on the σ-algebra of Borel subsets of $\mathbb{R}$.
  (b) Show that $\int_{\mathbb{R}} g \,\mathrm{d}(f_*\mu) = \int_X g \circ f \,\mathrm{d}\mu$ for every Borel function $g\colon \mathbb{R} \to [0,\infty]$.

(4) (a) State without proof the dominated convergence theorem.
  (b) Define $f_n\colon (0,\infty) \to \mathbb{R}$ by $f_n(x) = \frac{1}{x^{3/2}} \sin(\frac{x}{n})$. Compute the limit $\lim_{n\to\infty} \int_0^\infty f_n(x)\,\mathrm{d}x$.
  You must fully justify your argument.
  Hint: for small t the bound $|\sin(t)| \le t$ may be useful.

Part 2. Complex Analysis

(1) Evaluate $\int_\infty^\infty \frac{1}{x^6+1} dx$.

(2) Determine $z \in \mathbb{C}$, $r > 0$ so that $0, 1, 2+i \in \partial B_r(z)$.

(3) State and prove the Cauchy integral formula.

(4) The fundamental theorem of algebra states that any polynomial

$$p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0$$

has exactly n complex zeros, counting multiplicities.
Prove this by comparing two appropriate polynomials using Rouché's theorem.