Part 1. Real analysis

1. (a) Let $(X, A, \mu)$ be a measure space and $E_n \in A$ be measurable sets for $n \in \mathbb{N}$. Prove that the set is measurable:

$$E = \{x: x \text{ belongs to exactly two of the sets } E_n\}$$

(b) Suppose the measure space is a probability measure space (i.e., has measure 1) and every $x$ is contained in exactly 3 of the $E_n$’s. Compute $\Sigma \mu(E_n)$. State what tools you are using. Hint: $\mu E = \int \chi_E$.

2. (a) State the Radon-Nikodym theorem and define the Radon-Nikodym derivative.

(b) Show that if $(X, A, \nu)$ is a measure space, $g$ is an integrable function and $\int_E g \, d\nu = 0$ for every measurable set $E$, then $g = 0 \ \nu$-almost everywhere.

(c) Show for the case of finite measures that the Radon-Nikodym derivative is unique.

3. (a) Define what it means for a function $f: [a, b] \to \mathbb{R}$ to be absolutely continuous.

(b) Prove that the product of two absolutely continuous functions is absolutely continuous.

(c) Prove that if $f, g: [a, b] \to \mathbb{R}$ are absolutely continuous functions then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'.$$

4. Calculate

$$\int_0^1 \int_0^\infty y \sin xe^{-xy} \, dx \, dy.$$  

You must fully justify your solution.  

Hint: First use integration by parts twice to evaluate $\int_0^\infty \sin xe^{-xy} \, dx$.

Part 2. Complex Analysis

1. (a) Sketch the trajectory of the path $\gamma$ defined by $\gamma(t) = 1 + it + t^2$, $0 \leq t \leq 1$.

(b) Compute

$$\int_\gamma (\overline{z} + e^z) \, dz.$$  

2. Show there is exactly one complex number $z$ with $|z| < 1$ satisfying

$$e^z - 4z - 1 = 0.$$  

Hint: Rouche’s Theorem.

3. Evaluate $\int_0^\infty \frac{\cos(x)}{(1+x^2)^2} \, dx$.

4. Suppose $f$ is an entire function and $|f(z)| \leq A|z|^N + B$ for all $z$ where $A, B \in \mathbb{R}$. Show $f$ is a polynomial.