# REAL & COMPLEX ANALYSIS PRELIMINARY EXAMINATION
# FALL 2016, MATHEMATICS, UNIVERSITY OF CINCINNATI

## Part 1. Real analysis

(1) (a) Let $(X, \mathcal{A}, \mu)$ be a measure space and $E_n \in \mathcal{A}$ be measurable sets for $n \in \mathbb{N}$.
Prove that the set is measurable:

$$E = \{x : x \text{ belongs to exactly two of the sets } E_n\}$$

(b) Suppose the measure space is a probability measure space (i.e., has measure 1) and every $x$ is contained in exactly 3 of the $E_n$'s. Compute $\Sigma\mu(E_n)$. State what tools you are using.
Hint: $\mu E = \int \chi_E$.

(2) (a) State the Radon-Nikodym theorem and define the Radon-Nikodym derivative.
(b) Show that if $(X, \mathcal{A}, \nu)$ is a measure space, $g$ is an integrable function and $\int_E g \, \mathrm{d}\nu = 0$ for every measurable set $E$, then $g = 0$ $\nu$-almost everywhere.
(c) Show for the case of finite measures that the Radon-Nikodym derivative is unique.

(3) (a) Define what it means for a function $f : [a, b] \to \mathbb{R}$ to be absolutely continuous.
(b) Prove that the product of two absolutely continuous functions is absolutely continuous.
(c) Prove that if $f, g : [a, b] \to \mathbb{R}$ are absolutely continuous functions then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'.$$

(4) Calculate

$$\int_0^1 \int_0^\infty y \sin x e^{-xy} \, \mathrm{d}x \, \mathrm{d}y.$$

You must fully justify your solution.
Hint: First use integration by parts twice to evaluate $\int_0^\infty \sin x e^{-xy} \, \mathrm{d}x$.

## Part 2. Complex Analysis

(1) (a) Sketch the trajectory of the path $\gamma$ defined by $\gamma(t) = 1 + it + t^2$, $0 \le t \le 1$.

(b) Compute

$$\int_\gamma (\overline{z} + e^z) dz.$$

(2) Show there is exactly one complex number $z$ with $|z| < 1$ satisfying

$$e^z - 4z - 1 = 0.$$

Hint: Rouche's Theorem.

(3) Evaluate $\int_0^\infty \frac{\cos(x)}{(1+x^2)^2} dx$.

(4) Suppose $f$ is an entire function and $|f(z)| \le A\,|z|^N + B$ for all $z$ where $A, B \in \mathbb{R}$.
Show $f$ is a polynomial.