ANALYSIS PRELIMINARY EXAMINATION, FALL 2018

REAL ANALYSIS

In this part of the exam, m, dx, and dt denote Lebesgue measure on $\mathbb{R}$.

1. **(a)** Given two measures μ and ν on a σ-algebra $\mathcal{M}$ on a space X, define the notion $\mu \ll \nu$ (μ is absolutely continuous with respect to ν) and state the Radon-Nikodym theorem without proof.

(b) Does there exists a *finite* measure μ on the Lebesgue σ-algebra on $\mathbb{R}$ for which $\mu \ll m$ and also $m \ll \mu$? Either find such a measure or prove why that is impossible.

2. **(a)** Define the Lebesgue outer measure m^* on $\mathbb{R}$.

(b) Let $A \subset \mathbb{R}$ and assume that $m^*(A \cap (a,b)) \leq \frac{b-a}{2}$ for any $a, b \in \mathbb{R}$, $a < b$. Prove that A is Lebesgue measurable and $m(A) = 0$.

3. Let f be a Lebesgue integrable function on $[0,2]$ and for every $n \in \mathbb{N}$ and $x \in [0,1]$ define $f_n(x) := n \int_x^{x+\frac{1}{n}} f(t)dt$. Prove that $\int_0^1 |f_n(x) - f(x)|dt \to 0$ for $n \to \infty$. **Hint**: prove first the property for the case when f is continuous.

4. A Lipschitz function φ on a set $A \subset \mathbb{R}$ is a function such that there is a constant $L > 0$ for which $|\varphi(x) - \varphi(y)| \leq L|x-y|$ for all $x, y \in A$.

(a) Prove that every continuous piecewise linear function on $[0,1]$ is Lipschitz.

(b) Suppose that $f : [0,1] \to \mathbb{R}$ is an integrable function such that for every Lipschitz function φ on $[0,1]$ with $\varphi(0) = 0 = \varphi(1)$ we have $\int_0^1 f(x)\varphi(x)dx = 0$. Prove that $f = 0$ almost everywhere on $[0,1]$. **Hint**: recall that if g is integrable, then for every $\epsilon > 0$ there is a $\delta > 0$ such that if E is a measurable set and $m(E) < \delta$ then $\int_E |g|\, dm < \epsilon$.

COMPLEX ANALYSIS

In this part of the exam, $\mathbb{C}$ denotes the collection of all complex numbers.

1. In this question, Ω denotes a non-empty open subset of $\mathbb{C}$, and $f : \Omega \to \mathbb{C}$ has continuous first order partial derivatives in Ω. Let $a \in \Omega$.

(a) State a connection between f being complex differentiable at a and its first order partial derivatives satisfying the Cauchy-Riemann equations at a.

(b) Suppose that f is holomorphic in Ω, and let $g : \Omega \to \mathbb{C}$ be given by $g(z) = \overline{f(z)}$. Prove that g is complex differentiable at a if and only if $f'(a) = 0$.

2. Evaluate $\int_0^\infty \frac{1}{(1+x^2)^2}\, dx$ using the method of residues. Give justification for each of your steps in the solution.

3. **(a)** Let f be a holomorphic function on a non-empty open connected subset of $\mathbb{C}$. Prove that f is constant if $|f|$ is constant.

(b) Let u be a non-constant real-valued function on $\mathbb{C}$ that has continuous first partial derivatives in $\mathbb{C}$. Show that the function f given by $f(z) = \frac{u(z)-1-i}{u(z)-1+i}$ is not entire.

4. Find a conformal mapping of the plane that maps the real axis $\mathbb{R}$ to the unit circle centered at 0 and the imaginary axis $i\mathbb{R}$ to itself, and find the image of the unit circle centered at 0 under this map.