PhD Preliminary Exam in Algebra and Topology
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Full credit can be obtained by complete answers to 5 questions, of which at least two must come from each section. The examination lasts four hours.

Algebra

(1) Let $K \subset L$ be a finite field extension.
  (a) Prove that if $[L : K] = 2$, then the extension $K \subset L$ is normal.
  (b) Prove or give a counterexample: if $[L : K]$ is prime, then the extension $K \subset L$ is normal.
  (c) Prove or give a counterexample: if $[L : K] = 2$ is prime, then the extension $K \subset L$ is separable.

(2) Let k be a field and let $R = k[x^2, x^3]$ denote the subring of the polynomial ring $k[x]$ generated by k and x^2 and x^3. Prove that every ideal of R can be generated by two elements. Hint: if the ideal is nonzero, we may choose one of the generators to be a polynomial of least degree.

(3) Let $f(X) \in Q[X]$ be a polynomial of degree 5, and let K be a splitting field of f over $\mathbb{Q}$. Suppose that $\mathrm{Gal}(K/Q)$ is the symmetric group S_5.
  (a) Show that f is irreducible over $\mathbb{Q}$.
  (b) If α is a root of f, show that the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.
  (c) Show that $\alpha^5 \notin \mathbb{Q}$.

(4) Define what is meant by a Euclidean domain.
  (a) Prove that the ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean domain.
  (b) Prove that any Euclidean domain is a principal ideal domain.

Topology

(1) Let X be a metric space. Show that X is connected if and only if for every continuous map $f : X \to \mathbb{R}$, $f(X)$ is an interval.

(2) Suppose the topology of X is Hausdorff, and $f : X \to X$ is continuous. Show that the set $\{x \in X \mid f(x) = x\}$ is closed.

(3) Let $A = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 = 1,\ c = 0\}$ be the "equator" circle of S^2. Give an example of a map $f : S^2 \to S^2$ with the following properties:
   i) $f(x) \in A$ for all $x \in A$.
   ii) $f : S^2 \to S^2$ is homotopic to the identity map.
   iii) $f|_A : A \to A$ is not homotopic to the identity map.

(4) Let $p : E \to B$ be a covering map. We say a loop $\alpha : I \to B$ is *lift-preserved* if every path $\tilde{\alpha} : I \to E$ that satisfies $\alpha = p \circ \tilde{\alpha}$ is also a loop. Show that the set $\{[\alpha] \in \pi_1(B) :\ \alpha \text{ is lift-preserved}\} \subset \pi_1(B)$ is a normal subgroup of $\pi_1(B)$.