UNIVERSITY OF Cincinnati

Department of Mathematical Sciences
PO Box 210025
Cincinnati OH 45221-0025

4^{th} Floor French Hall West
Phone (513) 556-4050
Fax (513) 556-3417

MATHEMATICS QUALIFYING EXAM
AUGUST 2014

Four Hour Time Limit

$\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space

Proofs, or counter examples, are required for all problems.

(1) Let $\mathbb{R} \xrightarrow{f} \mathbb{R}$ be a continuous function.
  (a) Show that for each $x \in \mathbb{R}$,
$$f(x) - f(0) = \sum_{k=0}^{\infty} \left[f\left(\frac{x}{2^k}\right) - f\left(\frac{x}{2^{k+1}}\right) \right] .$$
  (b) Explain what it means to say that f is differentiable at $x = 0$.
  (c) Suppose that $\lim_{h \to 0} \dfrac{f(h) - f(h/2)}{h/2} = 0$. Use part (a), or some other method, to prove that f is differentiable at $x = 0$ with $f'(0) = 0$.

(2) Let $[0,1] \xrightarrow{f} \mathbb{R}$ be a continuous function.
  (a) Show that for each $\varepsilon \in (0,1)$,
$$\lim_{n \to +\infty} \int_0^{1-\varepsilon} f(x^n)\, dx = (1-\varepsilon)\, f(0).$$
  (b) Find
$$\lim_{n \to +\infty} \int_0^{1} f(x^n)\, dx.$$
  (Hint: Start by explaining why f is bounded.)

(3) Suppose that A and B are 3×3 matrices and AB is nonsingular. Prove that both A and B are nonsingular.

(4) Let m and n be positive integers with $m > n$. Prove that there do not exist $m \times n$ and $n \times m$ matrices A and B such that $AB = I_m$ (the $m \times m$ identity matrix).

(5) Let A be an $n \times n$ matrix whose entries are all real numbers. Suppose that A has n distinct non-zero eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ in $\mathbb{R}$. Let $\mathbf{v}_i \in \mathbb{R}^n$ be an eigenvector of A with corresponding eigenvalue λ_i. Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

(6) Let $[0,1] \xrightarrow{g} \mathbb{R}$ be a continuous function. Suppose that $(f_n)_1^\infty$ is a sequence of continuous functions $f_n : [0,1] \to [0,1]$ that converges uniformly on $[0,1]$ to some function $f : [0,1] \to [0,1]$. Prove that $(g \circ f_n)_1^\infty$ converges uniformly to $g \circ f$ on $[0,1]$.

Date: August 14, 2014.

(7) Define $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ by

$$f(x,y) := \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{when } (x,y) \neq 0\,, \\ 0 & \text{when } (x,y) = 0\,. \end{cases}$$

Prove that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ both exist, but f is not differentiable at $(0,0)$.

(8) Let $C := \{(x,y) \in \mathbb{R}^2 \mid (x+y)^3 = 3x + 5y\}$. Consider the point $p := (1,1)$ in C. Prove that there is an open neighborhood W of p such that $C \cap W$ is the graph $y = f(x)$ of some smooth function f defined near $x = 1$, and calculate $f'(1)$.