I. Algebra

(1). Let \( R \) be an integral domain.
(a) Define what it means for an element \( r \in R \) to be irreducible.
(b) Define what it means for an element \( r \in R \) to be prime.
(c) Show that in an integral domain a prime element is irreducible.
(d) Show that in a principal ideal domain that an irreducible element is prime.

(2). Suppose \( F = \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \), where \( \alpha_i^2 \in \mathbb{Q} \) for \( i = 1, 2, \ldots, n \). Prove that \( \sqrt[3]{3} \notin F \).

(3). Let \( F \) be a field.
(a) Let \( \alpha \in F \) be algebraic. Prove there is a unique monic irreducible polynomial \( m_\alpha(x) \in F[x] \) that has \( \alpha \) as a root.
(b) Prove that \( \alpha \in F \) is algebraic if and only if \( F(\alpha)/F \) is a finite extension.

(4). Let \( F = \mathbb{Q}(\sqrt[3]{3}, \sqrt{5})/\mathbb{Q} \).
(a) Prove that \( F \) is a Galois extension.
(b) Compute the Galois group.
(c) Explicitly give the correspondence between the subfields of \( F \) and the subgroups of the Galois group.

II. Topology

(1). Let \( X \) and \( Y \) be topological spaces and suppose \( f : X \to Y \). Show the following three conditions are equivalent:
(i) \( f \) is continuous.
(ii) For every subset \( A \subset X \), \( f(A) \subset f(A) \).
(iii) For every closed set \( B \subset Y \), the set \( f^{-1}(B) \) is closed in \( X \).

(2). Let \( A \) and \( B \) be subspaces of \( X \) and \( Y \), respectively. Let \( N \) be an open set in \( X \times Y \) containing \( A \times B \). Suppose \( A \) and \( B \) are compact. Show there exist open sets \( U \) and \( V \) in \( X \) and \( Y \), respectively, such that \( A \times B \subset U \times V \subset N \).

(3). Let \( X \) be a topological space.
(a) Show that if \( X \) is regular, every pair of points of \( X \) have neighborhoods whose closures are disjoint.
(b) Show that if \( X \) is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

(4). Let \( q : X \to Y \) and \( r : Y \to Z \) be covering maps. Set \( p := r \circ q \). Show that if \( r^{-1}(z) \) is finite for each \( z \in Z \), then \( p \) is a covering map.