Real Analysis
In this part of the exam, \( m \) or \( dx \) (resp., \( m^2 \)) denote Lebesgue measure on \( \mathbb{R} \) (resp., on \( \mathbb{R}^2 \)).

1. Carefully justifying your answer, evaluate:
\[
\lim_{n \to \infty} \int_0^\infty \frac{n \sin x}{1 + n^2 x^2} \, dx.
\]

2. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be a sequence of measurable functions. Show that the set
\[
\{ x \in \mathbb{R} : (f_n(x))_{n=1}^\infty \text{ converges to a real number} \}
\]
is measurable. Hint: a sequence in \( \mathbb{R} \) converges if and only if it is Cauchy.

3. Let \( f : [0, 1] \to \mathbb{R} \) be an absolutely continuous strictly increasing function. Prove that for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( E \subset [0, 1] \) and \( m^*(E) < \delta \), then \( m^*(f(E)) < \epsilon \), where \( m^* \) denotes the Lebesgue outer measure.

4. Let \( f \in L^1(0, \infty) \). For \( x > 0 \), define \( g(t, x) = tf(t)e^{-tx} \). Prove that \( g \in L^1((0, \infty) \times (0, \infty)) \) and
\[
\int_{(0,\infty) \times (0,\infty)} g(t, x) \, dm^2(t, x) = \int_0^\infty f(t) \, dm(t)
\]
justifying all your steps.

Complex Analysis
In this part of the exam, \( \mathbb{C} \) denotes the collection of all complex numbers.

1. Compute the following integral using the method of residues or the argument principle:
\[
\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 9)} \, dx.
\]

2. Let \( f \) be given by \( f(z) = \frac{1}{1+z} \) and for each positive integer \( n \) let the function \( g_n \) be the \( n \)-fold composition of \( f \) with itself, so \( g_2 = f \circ f \), \( g_3 = f \circ f \circ f \), etc.
   (a) Find an explicit formula for \( g_n \) for each positive integer \( n \).
   (b) Prove that the sequence \( g_n \) converges to zero uniformly on the disk \( \{ z : |z - 1| < 1 \} \).

3. Let \( a, b \in \mathbb{C} \) with \( a \neq b \), and let \( F(z) = \frac{z-a}{z-b} \).
   (a) Find the image of the line passing through \( a \) and \( b \) and \( \infty \).
   (b) Find the image of the perpendicular bisector of the line \( [a, b] \) (including \( \infty \) as a point in that line).
   (c) Find the image of the Euclidean circle centered at \( (a+b)/2 \) with radius \( |a - b|/2 \) (that is the circle centered at the midpoint between \( a \) and \( b \), and passing through both \( a \) and \( b \)).

4. Let \( f \) and \( g \) be two non-constant holomorphic (that is, complex analytic) functions in a region \( \Omega \subset \mathbb{C} \) such that \( |f(z)| \leq |g(z)| \) for all \( z \in \Omega \). Let \( K = g^{-1}(\{0\}) \). Prove that the function \( f/g \) is analytic on \( \Omega \setminus K \) and that it has an analytic extension to all of \( \Omega \). Use this to prove that if \( F \) is an holomorphic function on \( \mathbb{C} \) with \( |F(z)| \leq |\sin(\pi z)| \) for all \( z \in \mathbb{C} \) then there is some complex number \( c \) with \( |c| \leq 1 \) such that \( F(z) = c \sin(\pi z) \) for all \( z \in \mathbb{C} \).