1. Determine the largest element in the set \( \{1, \sqrt{2}, \sqrt[3]{3}, \ldots, \sqrt{n}\} \). Carefully justify all your claims.

2. Let \( \{f_n\}_{n=0}^{\infty} \) be a uniformly convergent sequence of bounded real functions on \( \mathbb{R} \). Show that the sequence \( \{f_n\}_{n=0}^{\infty} \) is uniformly bounded on \( \mathbb{R} \), i.e., there exists an \( M > 0 \) such that for all \( n \) and all \( x \in \mathbb{R} \), \( |f_n(x)| \leq M \).

3. Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of functions that map \([0, 1]\) into \( \mathbb{R} \) with the following properties:
   (a) For every \( n \), \( f_n(0) = 0 \).
   (b) For every \( n \), \( f_n(1) = 1 \).
   (c) For every \( n \), the function \( f_n \) is monotone increasing.
   (d) For every \( x \in (0, 1) \), \( \lim_{n \to \infty} f_n(x) = 1 \).
   Prove that \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 1 \).

4. Suppose that \( f \) is a continuous function on the interval \([0, 1]\) taking values in \([0, 2]\).
   Prove that there exists a number \( c \) in \([0, 1]\) such that \( f(c) = 2c \).
   Hint: Consider the function \( g(x) = f(x) - 2x \).

5. Consider the vector space \( V \) of all polynomials on interval \([0, 1]\), with the inner product on \( V \) given by
   \[ \langle p, q \rangle = \int_0^1 p(x)q(x) \, dx. \]
   Find an orthogonal basis for the linear subspace \( H \) of \( V \) consisting of all polynomials of degree less than or equal to 2. You must verify the orthogonality and the basis properties.

6. Let \( A, B \) be two \( n \times n \) matrices over \( \mathbb{R} \).
   (a) Show that if 0 is an eigenvalue of \( AB \), then 0 is also an eigenvalue of \( BA \).
   (b) Show that if \( \lambda \neq 0 \) is an eigenvalue of \( AB \), then \( \lambda \) is also an eigenvalue of \( BA \).

7. Define the trace of a matrix \( A \in \mathcal{M}_{n \times n}(\mathbb{R}) \) to be \( \text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} \), the sum of the diagonal entries of \( A \).
   (a) Prove that for \( A, B \in \mathcal{M}_{n \times n}(\mathbb{R}) \) we have \( \text{tr}(AB) = \text{tr}(BA) \).
   (b) Prove that if \( A = UCU^{-1} \) with \( U, C \in \mathcal{M}_{n \times n}(\mathbb{R}) \) and \( U \) invertible, then \( \text{tr}(A) = \text{tr}(C) \).

8. Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is differentiable with uniformly bounded partial derivatives
   \[ \left| \frac{\partial f_i}{\partial x_j}(x_1, x_2) \right| \leq 1 \text{ for } i, j = 1, 2 \text{ and all } x_1, x_2 \in \mathbb{R} \]
   Prove that there exists a constant \( L \) such that
   \[ \|f(x) - f(y)\| \leq L\|x - y\| \text{ for all } x, y \in \mathbb{R}^2 \]
   (Here \( f = (f_1, f_2) \) and \( x = (x_1, x_2) \).)