# MATHEMATICS QUALIFYING EXAM

## AUGUST 15, 2016

Four Hour Time Limit

**Notation:** $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is $n$-dimensional Euclidean space. The norm of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is $\|\mathbf{x}\| := \sqrt{\sum_{j=1}^n x_j^2}$. $\mathcal{M}_{m\times n}(\mathbb{K})$ is the set of $m$ by $n$ matrices over a field $\mathbb{K}$.

Unless explicitly stated, proofs, or counterexamples, are required for all problems.

1. Determine the largest element in the set $\{1, \sqrt{2}, \sqrt[3]{3}, \ldots, \sqrt[n]{n}\}$. Carefully justify all your claims.
2. Let $\{f_n\}_0^\infty$ be a uniformly convergent sequence of bounded real functions on $\mathbb{R}$. Show that the sequence $\{f_n\}_0^\infty$ is uniformly bounded on $\mathbb{R}$, i.e., there exists an $M > 0$ such that for all $n$ and all $x \in \mathbb{R}$, $|f_n(x)| \le M$.
3. Let $\{f_n\}_1^\infty$ be a sequence of functions that map $[0,1]$ into $\mathbb{R}$ with the following properties:
   (a) For every $n$, $f_n(0) = 0$.
   (b) For every $n$, $f_n(1) = 1$.
   (c) For every $n$, the function $f_n$ is monotone increasing.
   (d) For every $x \in (0,1)$, $\lim_{n\to\infty} f_n(x) = 1$.
   Prove that
   $$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1.$$
   (You may use without proof the fact that each $f_n$ is Riemann-integrable on $[0,1]$.)
4. Suppose that $f$ is a continuous function on the interval $[0,1]$ taking values in $[0,2]$. Prove that there exists a number $c$ in $[0,1]$ such that $f(c) = 2c$.
   Hint: Consider the function $g(x) = f(x) - 2x$.
5. Consider the vector space $V$ of all polynomials on interval $[0,1]$, with the inner product on $V$ given by $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$. Find an **orthogonal** basis for the linear subspace $H$ of $V$ consisting of all polynomials of degree less than or equal to 2. You must verify the orthogonality and the basis properties.
6. Let $A, B$ be two $n \times n$ matrices over $\mathbb{R}$.
   (a) Show that if 0 is an eigenvalue of $AB$, then 0 is also an eigenvalue of $BA$.
   (b) Show that if $\lambda \neq 0$ is an eigenvalue of $AB$, then $\lambda$ is also an eigenvalue of $BA$.
7. Define the trace of a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ to be $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$, the sum of the diagonal entries of $A$.
   (a) Prove that for $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
   (b) Prove that if $A = UCU^{-1}$ with $U, C \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $U$ invertible, then $\mathrm{tr}(A) = \mathrm{tr}(C)$.
8. Suppose $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with uniformly bounded partial derivatives
   $$\left|\frac{\partial f_i}{\partial x_j}(x_1, x_2)\right| \le 1 \text{ for } i, j = 1, 2 \text{ and all } x_1, x_2 \in \mathbb{R}$$
   Prove that there exists a constant $L$ such that
   $$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\| \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^2$$
   (Here $\mathbf{f} = (f_1, f_2)$ and $\mathbf{x} = (x_1, x_2)$).