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GRADUATE PROGRAM *QUALIFYING EXAM*

Four Hour Time Limit

$\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space

Proofs, or counter examples, are required for all problems.

(1) (a) Define what it means to say that a function $I \xrightarrow{f} \mathbb{R}$ is uniformly continuous; here $I \subset \mathbb{R}$ is an interval.
(b) Find an example of a function that is continuous at each point of an interval I but is not uniformly continuous on I. Be sure to prove that your function is not uniformly continuous.
(c) Give a condition, or conditions, on I that ensure that each continuous map $f : I \to \mathbb{R}$ is in fact uniformly continuous. (No proof required.)

(2) Let $(a_n), (b_n), (c_n)$ be sequences of real numbers with the property that for each $n \in \mathbb{N}$, $a_n \le b_n \le c_n$. Suppose that both series

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} c_n$$

converge. Prove that $\sum_{n=1}^{\infty} b_n$ converges.

(3) Let V be the vector space of all continuous functions $[-1, 1] \xrightarrow{f} \mathbb{R}$; here addition and scalar multiplication are defined as usual via $(f + g)(x) := f(x) + g(x)$ and $(c\,g)(x) := c\,g(x)$. Let W and Z be the collections of functions f in V that satisfy $f(-x) = -f(x)$ and $f(-x) = f(x)$, respectively, for all $x \in [-1, 1]$.
(a) Show that W is a vector subspace of V.
(b) Given that Z is also vector subspace of V, show that V is the direct sum of W and Z.

(4) Let $\mathbb{R}^2 \xrightarrow{L} \mathbb{R}^2$ be the linear map that does the following (in the order given):
(a) Triples the x component and doubles the y component.
(b) Rotates the resulting vector 45° clockwise around the origin.
(c) Projects the resulting vector onto the x-axis.
Write down the unique 2×2 matrix A that has the property that

$$\text{for all } \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2, \quad L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

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(5) Let V be the vector space of real symmetric $n \times n$ matrices.

(a) Show that

$$\langle A, B \rangle := \mathsf{tr}(AB)$$

defines an inner product on V, where $\mathsf{tr}(M)$ denotes the *trace* of a matrix M.

(b) Determine the dimensions of the following:

- V,
- the subspace W of V consisting of those matrices A such that $\mathsf{tr}(A) = 0$,
- the orthogonal complement $W^{\perp}$ of W in V (relative to the inner product defined in part (a)).

(6) Let $[a, b] \xrightarrow{f} \mathbb{R}$ be continuous. Prove that there exists a point $c \in [a, b]$ such that

$$f(c) \leq \frac{1}{2}\,[f(a) + f(b)]\ .$$

(7) Show that if f is differentiable, but unbounded on some finite interval (a, b), then f' is unbounded on (a, b). (Caution: f' need not be integrable!)

(8) (a) Define the notion of the *gradient* of a function $\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}$.

(b) Let f, g, h be the real-valued functions given by

$$f(x, y, z) := x^2 + yz\,,$$
$$g(x, y) := y^3 + xy\,,$$
$$h(x) := \sin(x)\,.$$

Compute the gradient of the function

$$\varphi(x, y, z) := h(f(x, y, z)g(x, y))\,.$$