1. Find the derivative of $f(x) = |x|^3$ (the answer should be written as a single formula).

**Solution:** Since $|x| = \sqrt{x^2}$ you can write $f(x) = |x|^3 = (x^2)^{3/2}$. The chain rule says that

$$f'(x) = \frac{3}{2}(x^2)^{1/2} \cdot 2x.$$  

So $f'(x) = 3x|x|$. 
2. Let \( f(x) \) be an infinitely differentiable periodic function of period \( 2\pi \) so that \( f(x + 2\pi) = f(x) \) for every \( x \). Show that

\[
\int_0^{2\pi} f'''(x)f(x) \, dx = 0.
\]

**Solution:** The periodicity of the function implies the periodicity of its derivatives as shown by differentiating the equation \( f(x + 2\pi) = f(x) \) as many times as we like. Integrate by parts using the periodicity of the function to eliminate boundary terms.

\[
\int_0^{2\pi} f'''(x)f(x) \, dx = - \int_0^{2\pi} f''(x)f'(x) \, dx = -\frac{1}{2} f^2(x) \bigg|_0^{2\pi} = 0.
\]
3. \[ \int_0^1 \frac{x}{x\sqrt{x} + 1} \, dx = ? \]

Hint: use substitution and partial fractions. Note that \( t = -1 \) is a root of the polynomial \( 1 + t^3 \).

**Solution:** Set \( u = \sqrt{x} \) and then integrate \( 2u^3/(u^3 + 1) \) by partial fractions.

To begin, set \( u = \sqrt{x} \) so that \( \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u} \). Then \( u \) goes from 0 to 1 as \( x \) does.

Since \( \frac{u^3}{u^3 + 1} = 1 - \frac{1}{(u+1)(u^2 - u + 1)} \) we write

\[
\frac{1}{(u + 1)(u^2 - u + 1)} = \frac{A}{u + 1} + \frac{Bu + C}{u^2 - u + 1}
\]

and require

\[
A + B = 0 \\
B + C - A = 0 \\
A + C = 1
\]

which says we want \( A = 1/3 \), \( B = -1/3 \) and \( C = 2/3 \).

\[
2 \int_0^1 \frac{u^3}{u^3 + 1} \, du = 2 \int_0^1 \left[ 1 - \frac{1/3}{u + 1} + \frac{(-1/3)u + 2/3}{u^2 - u + 1} \right] \, du
\]

\[
= 2 \left( u - \frac{1}{3} \log(u + 1) \right) \bigg|_0^1 + \int_0^1 \frac{2}{3} \frac{u - 2}{u^2 - u + 1} \, du
\]

\[
= 2(1 - \log(2)/3) + \frac{2}{3} \int_0^1 \frac{u - 2}{u^2 - u + 1} \, du
\]

In the remaining integral, write \( v = u^2 - u + 1 \) so that \( dv = 2u - 1 \) and

\[
\int \frac{u - 2}{u^2 - u + 1} \, du = \frac{1}{2} \int \frac{2u - 4}{u^2 - u + 1} \, du
\]

\[
= \frac{1}{2} \int \frac{dv}{v} - \frac{3}{2} \int \frac{1}{u^2 - u + 1} \, du
\]

\[
= \frac{1}{2} \log(u^2 - u + 1) - \frac{3}{2} \int \frac{1}{(x - 1/2)^2 + 3/4}.
\]

The remaining antiderivative is an arctan. In all we get the the antiderivative

\[
\frac{1}{3} \left( 6\sqrt{x} - 2\log(\sqrt{x} + 1) + \log(x - \sqrt{x} + 1) - 2\sqrt{3} \arctan(\frac{2}{\sqrt{3}}(\sqrt{x} - 1/2)) \right).
\]
4. Assume that $a > -1$ and $b > -1$. Calculate the limit
\[
\lim_{n \to \infty} \frac{n^b - a^1 + 2^a + \cdots + n^a}{1^b + 2^b + \cdots + n^b}.
\]

Hint: Look for Riemann sums in the expression.

**Solution:** The expression in question can be written as
\[
\frac{(1/n)^a + (2/n)^a + \cdots + (n/n)^a}{(1/n)^b + (2/n)^b + \cdots + (n/n)^b}.
\]

Multiplying top and bottom by $1/n$ we see that we have right Riemann sums for $\int_0^1 x^a \, dx$ and $\int_0^1 x^b \, dx$ both of which are convergent (if they’re improper) because $a, b > -1$. So the limit of the ratio is the ratio of the limits, namely
\[
\frac{\int_0^1 x^a \, dx}{\int_0^1 x^b \, dx} = \frac{b + 1}{a + 1}.
\]
5. (a) Show that the infinite series \( \sum_{n=0}^{\infty} ne^{-n} \) converges.

(b) Compute the sum of the series from part (a). Hint: What is the sum \( \sum x^n \) and how is it related to the sum \( \sum nx^n \)?

**Solution:** For \( |x| < 1 \) we have that the geometric series, \( \sum_{0}^{\infty} x^n = \frac{1}{1-x} \), is absolutely convergent. This tells us that, on the same interval, \( \sum n x^{n-1} = \frac{1}{(1-x)^2} \) and so, \( \sum_{0}^{\infty} nx^n = \frac{x}{(1-x)^2} \).

So \( \sum_{n=0}^{\infty} ne^{-n} = \frac{x}{(1-x)^2} \bigg|_{x=1/e} \).
6. The centers of two disks of radius 1 are one unit apart. Find the area of the union of the two disks.

**Solution:**

The area of the overlap of the two disks is made up of 4 congruent regions each of which has the area of a sector (1/6 of one disk) less half the area of an equilateral triangle with side length 1.

$$\left| D_1 \cup D_2 \right| = |D_1| + |D_2| - |D_1 \cap D_2|$$

$$= 2\pi - 4\left(\frac{\pi}{6} - \frac{1}{4} \cdot 1 \cdot \frac{\sqrt{3}}{2}\right).$$

If you really want to do an integral, you can. The region making 1/4 the overlap of the disks (shown in the figure) can be described in polar coordinates as $\sec(\theta)/2 < r < 1$ for $0 < \theta < \pi/3$ so its area can be written as

$$(1/2) \int_{\theta_1}^{\theta_2} [r_2^2 - r_1^2] \, d\theta = \frac{1}{2} \int_{0}^{\pi/3} 1 - \frac{1}{4} \sec^2(\theta) \, d\theta.$$

Then the hardest thing is to recall that the derivative of tan is $\sec^2$. 
7. Compute the limit as $t \to \infty$ of the average value of the function $f(x) = (1 - 2/x)^x$ on the interval $[2, t]$. Show that any rules or laws you use do, in fact, apply.

Solution:
We must evaluate

$$\lim_{x \to \infty} \frac{1}{x-2} \int_2^x (1 - 2/t)^t \, dt.$$ 

Since (using L'Hospital for the ratio has the form $0/0$ in line 2 below)

$$\lim_{x \to \infty} (1 - 2/x)^x = \exp \left( \lim_{x \to \infty} x \log(1 - 2/x) \right)$$

$$= \exp \left( \lim_{x \to \infty} \frac{\log(1 - 2/x)}{1/x} \right)$$

$$= \exp \left( \lim_{x \to \infty} \frac{-1/x^2}{-2} \right)$$

$$= \exp \left( -2 \lim_{x \to \infty} \frac{1}{1 - 2/x} \right)$$

$$= \exp(-2),$$

we see that $\int_2^\infty f(x) \, dx$ diverges because the integrand has non-zero limit. Therefore we may apply L'Hospital to the limit of the average value which has form $\infty/\infty$.

We find, using the fundamental theorem of calculus, that

$$\lim_{x \to \infty} \frac{\int_2^x (1 - 2/t)^t \, dt}{x-2} = \lim_{x \to \infty} (1 - 2/x)^x = e^{-2}.$$