Preliminary Exam
Differential Equations
August 16, 2017

Name: Student Id #: 

Instruction: Do all eight problems.

Score:

Problem 1.1 ————- Problem 2.1————-
Problem 1.2 ————- Problem 2.2————-
Problem 1.3————— Problem 2.3————-
Problem 1.4————— Problem 2.4————-

Part I total score : ———- Part II total score —————-

Total score ——————
Problem 1.1

Let $A$ be an invertible $3 \times 3$ matrix, and consider the equation $x'(t) = Ax(t)$. Suppose there are three solutions $x(t)$, $y(t)$, $z(t)$ with the properties

- $\lim_{t \to \infty} x(t) = 0$.
- $\lim_{t \to -\infty} y(t) = 0$.
- $z(4\pi) = z(0)$.

Show that at least one of $x(t)$, $y(t)$, $z(t)$ must be the constant solution at the origin.
Problem 1.2

The system of equations \[
\begin{cases}
x' = x + 3 \sin y \\
y' = x^2 + 4x - 3y
\end{cases}
\]
has an equilibrium point at the origin \(x = 0, y = 0\).

Determine whether the equilibrium is asymptotically stable, stable, or unstable.
Problem 1.3

Use the appropriate Lyapunov function to determine the stability of the equilibrium point of the system

\[
\begin{align*}
\dot{x}_1 &= -2x_2 + x_2x_3 \\
\dot{x}_2 &= x_1 - x_1x_3 \\
\dot{x}_3 &= x_1x_2
\end{align*}
\]
Problem 1.4

Consider the autonomous differential equation

\[ v_{xx} + v - v^3 + v_0 = 0 \]

in which \( v_0 \) is a constant.

a) Show that for \( v_0^2 < \frac{4}{27} \), this equation has 3 stationary points and classify their type.

b) For \( v_0 = 0 \), draw the phase plane for this equation.
Problem 2.1.

Solve the following initial value problem.

\[ u_x^2 + yu_y - u = 0 \text{ with the initial condition } u(x, 1) = 1 + x^2/4. \]
Problem 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. Consider a non-linear boundary value problem ($u \in C^{1,1}(\Omega)$)

\[
\begin{cases}
-\Delta u + \kappa_{(u>0)} = 0 \text{ in } \Omega \\
u = \phi \text{ on } \partial \Omega
\end{cases}
\]

where

\[
\kappa_{(u>0)}(x) = \begin{cases}
1 \text{ if } u(x) > 0, \\
0 \text{ if } u(x) \leq 0.
\end{cases}
\]

Prove that $u(x) \geq 0$ in $\Omega$ if $\phi > 0$ on $\partial \Omega$. 

Problem 2.3.

(i) Show that if a function $u \in C(\Omega)$ satisfies the mean value property for each ball $B(x, r) \subset \Omega$, then $u \in C^\infty(\Omega)$.

(ii) Let $\{u_n\}_{n=1}^\infty$ be a sequence of harmonic functions in $C(\Omega)$. If $u_n$ is uniformly convergent to $u$ in $\Omega$ as $n \to \infty$, then $u$ is also harmonic function in $\Omega$. 
Problem 2.4.

Fix a number $L > 0$ and consider the initial-boundary value problem of the linear six-order Boussinesq equation

$$\begin{cases} 
    u_{tt} - u_{xx} + u_{xxxx} - u_{xxxxxx} = 0 \text{ in } (0, L) \times (0, T), \\
    u(x, 0) = g(x) \text{ and } u_t(x, 0) = h(x), \\
    u(0, t) = 0, \ u(L, t) = 0, \ u_{xx}(0, t) = 0, \ u_{xx}(L, t) = 0, \ u_{xxxx}(0, t) = 0, \ u_{xxxx}(L, 0) = 0
\end{cases} \tag{*}
$$

i) Define $E(t) = \int_0^L \left( u_t^2(x, t) + u_x^2(x, t) + u_{xx}^2(x, t) + u_{xxx}^2(x, t) \right) dx$. Show that

$$E(t) = \int_0^L \left( h^2(x) + (g'(x))^2 + (g''(x))^2 + (g'''(x))^2 \right) dx$$

for any $0 \leq t \leq T$.

ii) Show that (*) admits at most one smooth solution.