Real Analysis
In this part of the exam, $m, \, dx, \, dt$ denote the Lebesgue measure on $\mathbb{R}$.

1. Let $f$ be a monotone increasing absolutely continuous function on $[0,1]$. Prove that for every open set $\emptyset \subset (f(0), f(1))$, the integral $\int_{f^{-1}(\emptyset)} f'(x) \, dx$ exists and $\int_{f^{-1}(\emptyset)} f'(x) \, dx = m(\emptyset)$. \textbf{Hint} Notice that $f$ needs not be strictly increasing.

2. Let $f \in L^1(\mathbb{R})$ and for every $x \in \mathbb{R}$ let $F(x) = \int_x^{x+1} f(t) dt$.
   (a) Prove that $F$ is absolutely continuous on every interval $[a,b]$ and in particular $F$ is measurable. \textbf{Hint} Tonelli.
   (b) Compute $\int_{\mathbb{R}} F(x) \, dx$.

3. (a) Prove that if $f \geq 0$ is a Riemann integrable function on every interval $[0,a]$ with $0 < a < \infty$ and if the improper Riemann integral $\int_0^\infty f(x) \, dx$ converges, then $f \in L^1([0,\infty))$ and its Lebesgue integral coincides with the improper Riemann integral.
   (b) Evaluate $\lim_{n \to \infty} \int_0^\infty (1 + \frac{z}{n})^{-2} \, dx$. \textbf{Hint} $1 + y \leq e^y$ for every $y \in \mathbb{R}$.

4. Let $f_n$ be a sequence of Lebesgue measurable functions on $\mathbb{R}$ converging a.e. to a function $f$.
   (a) Assume that $0 \leq f_n(x) \leq f(x)$ a.e. Does it follow that $\int_{\mathbb{R}} f_n \, dm \to \int_{\mathbb{R}} f \, dm$? Either prove or find a counterexample. Does the answer change if we replace the condition that $f_n(x) \geq 0$ a.e., with the condition that $f_n$ is integrable for all $n$?
   (b) Assume that $f_n(x) \geq f(x) \geq 0$ a.e. Does it follow that $\int_{\mathbb{R}} f_n \, dm \to \int_{\mathbb{R}} f \, dm$? Either prove or find a counterexample. Does the answer change if the integral is over $[0,1]$ rather than $\mathbb{R}$?

Complex Analysis
In this part of the exam, $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{D}$ denotes the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. The unit circle centered at 0 is denoted $\mathbb{T}$. For complex-valued functions on a planar domain, the term “holomorphic” is the same as “analytic”, and means a complex differentiable function.

1. Find a conformal map that maps the planar domain
   $$\Omega := \{ z \in \mathbb{C} : |z| > 1 \text{ and } \text{Re}(z) > 0 \}$$
to the unit disk $\mathbb{D}$.

2. For $0 \neq z \in \mathbb{C}$ let $f(z) = e^{1/z}$.
   (a) For $r > 0$ let $\gamma_r$ be the counterclockwise oriented unit circle $\{ z \in \mathbb{C} : |z| = r \}$ centered at 0.
      Show that the function $\varphi : (0,\infty) \to \mathbb{C}$ given by
      $$\varphi(r) = \int_{\gamma_r} f(z) \, dz$$
is independent of $r$.
   (b) Show that $\lim_{z \to \infty} z \, f(z) = 1$.
   (c) Compute $\varphi(r)$ for $r > 0$.

3. Let $\Omega$ be a simply connected planar domain and $f : \Omega \to \mathbb{C}$ be a holomorphic function such that whenever $\gamma$ is an anticlockwise oriented piecewise smooth loop in $\Omega$ we have
   $$\int_{\gamma} \frac{f'(w)}{f(w)} \, dw = 0.$$
   Show that there is a holomorphic function $g : \Omega \to \mathbb{C}$ such that $e^{\varphi(z)} = f(z)$ for each $z \in \Omega$. \textbf{Hint:}
   Fix a point $z_0 \in \Omega$ and consider the function $g : \Omega \to \mathbb{C}$ given by $g(z) = \int_{\gamma_z} \frac{f'(w)}{f(w)} \, dw$, where $\gamma_z$ is any choice of a piecewise smooth curve in $\Omega$ with starting point $z_0$ and end point $z$. If you use this hint, you should also explain why this function $g$ is independent of the choice of $\gamma_z$.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic, and suppose that $f(\mathbb{T}) \subset \mathbb{R}$.
   (a) Show that $f(0)$ is a real number.
   (b) Show that for each $z \in \mathbb{D}$ we must have $f(z) \in \mathbb{R}$.
   (c) Show that $f$ must be a constant function.