1. Proof or counter-example: If $f$ is an odd, differentiable function defined for all $x$ then $f'$ is an even function.

**Solution:** $f$ is odd exactly when $-f(-x) = f(x)$ for all $x$. Differentiating, (using the chain rule!) this says

$$f'(-x) = f'(x)$$

which says $f'$ is an even function.
2. Evaluate $\int_{0}^{\pi/2} \sqrt{\sin(x) + 1} \, dx$

**Solution:** One approach is to write $\sin x = \sin 2(x/2) = 2 \sin(x/2) \cos(x/2)$ so that

$$1 + \sin x = \cos^2(x/2) + \sin^2(x/2) + 2 \sin(x/2) \cos(x/2)$$

$$= (\cos(x/2) + \sin(x/2))^2$$

Since $\cos(x/2) + \sin(x/2) \geq 0$ on the domain of integration

$$\sqrt{(\cos(x/2) + \sin(x/2))^2} = |\cos(x/2) + \sin(x/2)| = \cos(x/2) + \sin(x/2),$$

so the integral becomes
3. Find a function $f(x)$ differentiable for all $x \neq 0$ that satisfies

1. $f'(x) = -\frac{10}{x^2}$ for $x \neq 0$
2. $f(-2) = 3$
3. $f(1) = -2$

**Solution:** Finding an antiderivative isn’t hard. The point is that different constants of integration can be used on different intervals of continuity. So this works:

$$f(x) = \begin{cases} 
\frac{10}{x} + 8 & \text{if } x < 0 \\
\frac{10}{x} - 12 & \text{if } x > 0
\end{cases}$$
4. Find all values of $k$ for which the limit exists:

\[ \lim_{x \to 3} \frac{4x^2 + kx + 7k - 6}{2x^2 - 5x - 3}. \]

List all possible values of $k$, find the limit for each such $k$, and explain why no other $k$’s exist.

Solution: Unless the polynomial in the numerator of this rational function vanishes at $x = 3$ the limit of the numerator is nonzero and of the denominator is 0 as $x \to 3$ — so the entire limit doesn’t exist. So the only values of $k$ for which the limit can exist are those for which

\[ 4(3^2) + 3k + 7k - 6 = 0, \]

That is, $k = -3$. In this case

\[ \lim_{x \to 3} \frac{4x^2 - 3x - 27}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{(x - 3)(4x + 9)}{(x - 3)(2x + 1)} \]

\[ = \lim_{x \to 3} \frac{4x + 9}{2x + 1} \]

\[ = 21/7 = 3. \]
5. Find the limit or prove that it doesn’t exist:

$$\lim_{N \to \infty} \sum_{1 \leq n \leq N} \frac{1}{n \log(N)}$$

**Solution:** The limit exists and equals 1.

Comparing the functions $1/(x + 1)$ and $1/x$ with the step function taking the values $1/n$ on the intervals $[n - 1, n)$ for $n = 1, 2, 3, \cdots$ we see that

$$\int_{0}^{n} \frac{1}{x + 1} \, dx \leq \sum_{1 \leq k \leq n} \frac{1}{k} \leq 1 + \int_{1}^{n} \frac{1}{x} \, dx$$

so that

$$\log(n + 1) \leq \sum_{1 \leq k \leq n} \frac{1}{k} \leq 1 + \log(n).$$

Therefore

$$\frac{\log(n + 1)}{\log(n)} \leq \frac{\sum_{1 \leq k \leq n} \frac{1}{k}}{\log(n)} \leq 1 + \frac{1}{\log n}.$$ 

The expressions on the left and right in this inequality both have limit 1 as $n \to \infty$. 
6. Let \( w(x) \) be a positive-valued function on the interval \([a, b]\), and suppose that \( f(x) \) is a continuous function on \([a, b]\). Define the weighted-average \( u \) of \( f(x) \) over the interval \([a, b]\) by

\[
u = \frac{\int_a^b w(x)f(x) \, dx}{\int_a^b w(x) \, dx}.
\]

Show that there exists \( c \) in \((a, b)\) satisfying \( f(c) = u \).

**Hint:** \( f \) is continuous on \([a, b]\) so it attains a minimum value \( m \) and a maximum value \( M \) on \([a, b]\), satisfying

\[
m \leq f(x) \leq M.
\]

Start with multiplying this inequality by an appropriate function and remember to use the Intermediate Value Theorem when the time comes!

**Solution:** The intermediate value theorem assures us that for every \( y \) with \( m \leq y \leq M \) there is a \( c \) between \( a \) and \( b \) so that \( f(c) = y \).

Since \( w > 0 \) we can calculate as follows

\[
m \leq f(x) \leq M \\
w(x)m \leq w(x)f(x) \leq w(x)M \\
m \int_a^b w(x) \, dx \leq \int_a^b w(x)f(x) \, dx \leq M \int_a^b w(x) \, dx \\
m \leq \frac{\int_a^b w(x)f(x) \, dx}{\int_a^b w(x) \, dx} \leq M
\]

And so there is a \( c \) between \( a \) and \( b \) (inclusive) for which

\[
f(c) = \frac{\int_a^b w(x)f(x) \, dx}{\int_a^b w(x) \, dx}.
\]
7. Compute the limit

\[ \lim_{{n \to \infty}} \frac{1}{n} \sum_{{k=1}}^{n} \arctan\left(\frac{k}{n}\right) \]

**Solution:** Notice that the sums in question are Riemann sums with uniform partitions for a definite integral with integrand \( \arctan(x) \). So the limit is \( \int_{0}^{1} \arctan(x) \, dx \). You can evaluate this by finding an antiderivative with one application of integration by parts.