1. Find $\frac{d}{dx}(f(x))$ if $\frac{d}{dx}(f(3x)) = 6x$.

**Solution:** Answer: $f'(x) = \frac{2x}{3}$.

Set $u = 3x$. Then $f'(u) = \frac{2}{3}u$, so $f'(x) = \frac{2}{3}x$.

2. Find a polynomial $f(x)$ with the property that $f(-1)$ and $f(1)$ are local maxima and $f(0) = -1$ is a local minimum.

**Solution:** Require that $f'(-1) = f'(0) = f'(1)$, that $f' > 0$ for $x < -1$ and $0 < x < 1$ while $f' < 0$ for $-1 < x < 0$ and $x > 1$.

One such function is $f'(x) = -x(x+1)(x-1)$. In that case $f(x) = x^2/2 - x^4/4$ has the required extremal properties and $f(x) = x^2/2 - x^4/4 - 1$ also has $f(0) = -1$.

3. What is the maximum value of $g(x) = |\sin(x) - 2\cos(x)|^2$?

**Solution:** Answer: 5.

Pick $\theta$ in $(0, \pi/2)$ so that $\cos(\theta) = 1/\sqrt{5}$. Then $\sin(\theta) = 2/\sqrt{5}$ and

$$
\sin(x) - 2\cos(x) = \sqrt{5}(\sin(x)\cos(\theta) - \cos(x)\sin(\theta)) \\
= \sqrt{5}\sin(x - \theta)
$$

Therefore

$$
g(x) = 5\sin^2(x - \theta)
$$

from which we see that the maximum value of $g$ is 5 since $\sin^2$ assumes its maximum value 1.

4. For what values of the constant $t$ does the function

$$
F(x) = (x^2 + t)e^x
$$

have two distinct inflection points?
Solution: Answer: $t < 2$. 

$F'' = (x^2 + 4x + 2 + t)e^x$ the requirement that $F''$ have two distinct roots reduces to $16 - 4(2 + t) > 0$ and that says $t < 2$.

5. Evaluate

$$\int_0^1 \sqrt[6]{1-x^6} - \sqrt[8]{1-x^8} \, dx$$

Hint: what is the inverse of the function $y = (1-x)^b$?

Solution: Answer:

$$\int_0^1 \sqrt[6]{1-x^6} - \sqrt[8]{1-x^8} \, dx = 0.$$ 

The functions $\sqrt[6]{1-x^8}$ and $\sqrt[8]{1-x^6}$ are both monotone decreasing on $0 \leq x \leq 1$ and each is the inverse of the other. That means their graphs are reflections of each other in the line $y = x$ as are the areas under the graph of one of them and to the left of the graph of the other. These areas are represented by $\int_0^1 \sqrt[6]{1-x^8} \, dx$ and $\int_0^1 \sqrt[8]{1-x^6} \, dx$. Reflected areas are congruent. So the integral over $[0,1]$ of the difference of the functions is 0.

You could try to actually evaluate the antiderivative and that’d be a fine, if more time consuming, approach. And, you’d need to know facts about hypergeometric functions. Perhaps easier is the following: start by evaluating

$\int_0^1 \sqrt[6]{1-x^8} \, dx$ by substitution setting $y = y(x) = \sqrt[8]{1-x^8}$. Then $y(0) = 1$ and $y(1) = 0$ and $x(y) = \sqrt[6]{1-y^6}$ so (integrating by substitution and then integrating by parts)

$$\int_0^1 \sqrt[6]{1-x^8} \, dx = \int_1^0 y(x)x'(y) \, dy$$

$$= yx(y)\bigg|_1^0 - \int_1^0 x(y) \, dy$$

$$= 0 - \int_1^0 x(y) \, dy$$

$$= \int_0^1 \sqrt[6]{1-y^6} \, dy$$
6. For which values of \( t \) is the series

\[
\sum_{n \geq 1} n^{1/n^t} - 1
\]

convergent?

**Solution:** Answer: The series converges if and only if \( t > 1 \).

Write the sum as \( \sum_{n \geq 1} e^{\frac{\log(n)}{n^t}} - 1 \).

Since \( \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \) the limit comparison test says it is sufficient to consider the series

\[
\sum_{n \geq 1} \frac{\log(n)}{n^t}.
\]

If \( t \leq 1 \) then \( \log(n)/n^t > 1/n \) for large \( n \) and the series diverges by comparing with the harmonic series.

It \( t > 1 \) then for large \( n \) we have \( \log(n) < n^{\frac{t-1}{2}} \) so that for large \( n \),

\[
\frac{\log(n)}{n^t} < \frac{1}{n^{(t+1)/2}}.
\]

And then, because \((t+1)/2 > 1\), the series

\[
\sum_{1} \frac{1}{n^{(t+1)/2}}
\]

converges.

7. Of all the parallelograms \( ABCD \) having \( A \) and \( C \) on the \( y \)-axis, having \( B \) and \( D \) on the \( x \)-axis and containing the ellipse \( x^2/2 + y^2/3 = 1 \), which has the smallest area?
Solution: Answer: the one with vertices \((\pm 2,0), (0, \pm \sqrt{6})\) and area \(A = 4\sqrt{6}\).

Due to the symmetry of the ellipse with respect to the \(x, y\), axes it is enough to work in the first quadrant and minimize the area of a right triangle with 90 angle at the origin. Clearly the triangle is minimized when the hypotenuse is tangent to the ellipse.

Setting

\[
\begin{align*}
u &= x/\sqrt{2} \\
v &= y/\sqrt{3}
\end{align*}
\]

gives the ellipse the equation \(u^2 + v^2 = 1\) and changes areas by

\[
\Delta x \Delta y = \sqrt{6} \Delta u \Delta v.
\]

At the point \((u, v) = (\cos \theta, \sin \theta)\) the tangent to the circle has slope \(-\cot \theta\) and equation

\[
v - \sin \theta = -\cot \theta (u - \cos \theta).
\]

The \(u\) and \(v\) intercepts of this line are \((\frac{1}{\cos \theta}, 0)\) and \((0, \frac{1}{\sin \theta})\) respectively so the area of the triangle is

\[
A = \frac{1}{2} \frac{1}{\sin \theta} \frac{1}{\cos \theta} = \frac{1}{\sin 2\theta}.
\]

The area is minimized when \(\sin 2\theta\) takes its maximal value — at \(\theta = \pi/4\) and the minimal value is \(A = 1\). The point of tangency is \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\).

In the \(x, y\) plane the intercepts are \(x = \sqrt{2}u = 2\) and \(y = \sqrt{3}v = \sqrt{6}\), and the minimal area is then \(A = \sqrt{6}\).

So the whole parallelogram we needed to find has vertices \((\pm 2,0), (0, \pm \sqrt{6})\) and area \(A = 4\sqrt{6}\).