Problem 1

Let \( X_1, X_2, \ldots, X_n \) be i.i.d from \( N(\mu, \sigma^2) \) population. Suppose that \( \sigma^2 = \lambda \mu^2 \) with unknown \( \lambda > 0 \) and \( \mu \in \mathbb{R} \). Find a Likelihood Ratio test for testing

\[ H_0 : \lambda = 1 \quad \text{vs.} \quad H_1 : \lambda \neq 1. \]

Problem 2

Suppose that \( \{ X_1, X_2, \ldots, X_n \} \) and \( \{ Y_1, Y_2, \ldots, Y_n \} \) are independent samples, \( X_i \)'s from \( \mathcal{E}(\lambda) \) population, and \( Y_j \)'s from \( \mathcal{E}(\mu) \) population; \( \lambda \) and \( \mu \) are unknown positive numbers. (The density function of \( X_1 \) is: \( f(x) = \frac{1}{\lambda} e^{-x/\lambda}, \ x > 0. \))

[2a] Find the UMVUE \( W \) of \( \theta = \frac{\mu}{\lambda} \).

[2b] Show that \( \frac{n}{n-1} \frac{\mu}{\lambda} W \) follows F-distribution.

Problem 3

Let \( X_1, X_2, \ldots, X_n \) be i.i.d. from Gamma distribution \( \text{Gamma}(k, \theta) \) with an unknown \( \theta > 0 \), and known \( k \). Let the prior be such that \( \varpi = \frac{1}{\theta} \) has the Gamma distribution \( \text{Gamma}(\alpha, \gamma) \) with known \( \alpha > 0 \) and \( \gamma > 0 \).

{ The density function of Gamma distribution \( \text{Gamma}(\alpha, \beta) \) is: \( f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \ x > 0 \) }

[3a] Under the square error loss, find the Bayes estimator of \( \theta \), \( T(X_1, X_2, \ldots, X_n) \).

[3b] Compute the bias of \( T(X_1, X_2, \ldots, X_n) \) as an estimator of \( \theta \).

Problem 4

If \( X_1, X_2, \ldots, X_n \) are iid \( U(0, \theta) \) random variables, it is known that the UMVE of \( \theta \) is

\[ \delta_n = \frac{n+1}{n} \max \{ X_{[n,n]} \} = \frac{n+1}{n} \max \{ X_1, X_2, \ldots, X_n \} \]

[4a] Determine the limit distribution of \( n(\theta - \delta_n) \) as \( n \to \infty \)

[4b] Find the limit \( \lim_{n \to \infty} \frac{E[X_{[n,n]} - \theta]^2}{E[\delta_n - \theta]^2} \).
Part II Probability

Problem 5. Let \( \{X_{nk} : k = 1, \ldots, n, n \in \mathbb{N} \} \) be a family of independent random variables satisfying
\[
P\left( X_{nk} = \frac{k}{\sqrt{n}} \right) = P\left( X_{nk} = -\frac{k}{\sqrt{n}} \right) = P(X_{nk} = 0) = 1/3
\]
Let \( S_n = X_{n1} + \cdots + X_{nn} \). Prove that \( S_n/s_n \) converges in distribution to a standard normal random variable for a suitable sequence of real numbers \( s_n \).

Some useful identities:
\[
\begin{align*}
\sum_{k=1}^{n} k &= \frac{1}{2}n(n+1) \\
\sum_{k=1}^{n} k^2 &= \frac{1}{6}n(n+1)(2n+1) \\
\sum_{k=1}^{n} k^3 &= \frac{1}{4}n^2(n+1)^2
\end{align*}
\]

Problem 6. Suppose \( \{X_n\} \) is a sequence of independent nonnegative random variables. Show that if \( \sum_{n=1}^{\infty} E(\min\{X_n, 1\}) < \infty \) then \( \sum_{n} X_n < \infty \) with probability one.

Problem 7. Let \( X_1, X_2, \ldots \) be independent and identically distributed random variables with \( E|X_1| = \infty \).

(a) Show that for each real number \( a > 0 \)
\[
\sum_{n=1}^{\infty} P(|X_n| \geq an) = \infty.
\]

(b) Show that
\[
\sup_{n} \frac{1}{n} |X_n| = \infty
\]
with probability one.

(c) Show that
\[
\sup_{n} \frac{1}{n} |X_1 + X_2 + \ldots + X_n| = \infty
\]
with probability one.

Problem 8. (a) Let \( X \) be a positive random variable. Show by a method of your choice that
\[
E(X) = \int_{0}^{\infty} P(X > x)dx.
\]

(b) Explain why
\[
\int_{0}^{\infty} P(X > x)dx = \int_{0}^{\infty} P(X \geq x)dx.
\]