MATHEMATICS QUALIFYING EXAM
AUGUST 2014
Four Hour Time Limit

\( \mathbb{R} \) is the field of real numbers and \( \mathbb{R}^n \) is \( n \)-dimensional Euclidean space.

Proofs, or counter examples, are required for all problems.

1. Let \( \mathbb{R} \xrightarrow{f} \mathbb{R} \) be a continuous function.
   (a) Show that for each \( x \in \mathbb{R} \),
   \[
   f(x) - f(0) = \sum_{k=0}^{\infty} \left[ f\left( \frac{x}{2^k} \right) - f\left( \frac{x}{2^{k+1}} \right) \right].
   \]
   (b) Explain what it means to say that \( f \) is differentiable at \( x = 0 \).
   (c) Suppose that \( \lim_{h \to 0} \frac{f(h) - f(h/2)}{h/2} = 0 \). Use part (a), or some other method, to prove that \( f \) is differentiable at \( x = 0 \) with \( f'(0) = 0 \).

2. Let \( [0, 1] \xrightarrow{g} \mathbb{R} \) be a continuous function.
   (a) Show that for each \( \varepsilon \in (0, 1) \),
   \[
   \lim_{n \to +\infty} \int_{0}^{1-\varepsilon} f(x^n) \, dx = (1 - \varepsilon) f(0).
   \]
   (b) Find
   \[
   \lim_{n \to +\infty} \int_{0}^{1} f(x^n) \, dx.
   \]
   (Hint: Start by explaining why \( f \) is bounded.)

3. Suppose that \( A \) and \( B \) are \( 3 \times 3 \) matrices and \( AB \) is nonsingular. Prove that both \( A \) and \( B \) are nonsingular.

4. Let \( m \) and \( n \) be positive integers with \( m > n \). Prove that there do not exist \( m \times n \) and \( n \times m \) matrices \( A \) and \( B \) such that \( AB = I_m \) (the \( m \times m \) identity matrix).

5. Let \( A \) be an \( n \times n \) matrix whose entries are all real numbers. Suppose that \( A \) has \( n \) distinct non-zero eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) in \( \mathbb{R} \). Let \( v_i \in \mathbb{R}^n \) be an eigenvector of \( A \) with corresponding eigenvalue \( \lambda_i \). Prove that \( v_1, \ldots, v_n \) are linearly independent.

6. Let \( [0, 1] \xrightarrow{g} \mathbb{R} \) be a continuous function. Suppose that \( (f_n)_{n=1}^{\infty} \) is a sequence of continuous functions \( f_n : [0, 1] \to [0, 1] \) that converges uniformly on \([0, 1]\) to some function \( f : [0, 1] \to [0, 1] \). Prove that \((g \circ f_n)_{n=1}^{\infty}\) converges uniformly to \( g \circ f \) on \([0, 1]\).
(7) Define $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ by
\[
f(x, y) := \begin{cases} 
    \frac{xy^2}{x^2 + y^2} & \text{when } (x, y) \neq 0, \\
    0 & \text{when } (x, y) = 0.
\end{cases}
\]
Prove that the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but $f$ is not differentiable at $(0, 0)$.

(8) Let $C := \{(x, y) \in \mathbb{R}^2 \mid (x + y)^3 = 3x + 5y\}$. Consider the point $p := (1, 1)$ in $C$. Prove that there is an open neighborhood $W$ of $p$ such that $C \cap W$ is the graph $y = f(x)$ of some smooth function $f$ defined near $x = 1$, and calculate $f'(1)$. 