QUALIFYING EXAM PRACTICE PROBLEMS

\( \mathbb{R} \) is the field of real numbers and \( \mathbb{R}^n \) is \( n \)-dimensional Euclidean space

Proofs, or counter examples, are required for all problems.

(1) Let \((X, d)\) be a metric space and \(S\) a subset of \(X\). State the logical implications that hold among the following conditions. (No proofs are required here, but where possible, provide ‘names’ of appropriate theorems.)

(a) \(S\) is bounded
(b) \(S\) is closed
(c) \(S\) is compact
(d) \(S\) is complete
(e) \(S\) is sequentially compact
(f) \(S\) is totally bounded

What changes if \(X = \mathbb{R}^n\) and \(d(x, y) = ||x - y||\)?

(2) Let \(x_n := (-1)^n \frac{\sqrt{n^2 + 1}}{n + 1}\). Is \((x_n)_{n=1}^{\infty}\) a Cauchy sequence?

(3) Determine \(\limsup_{n \to \infty} x_n\) and \(\liminf_{n \to \infty} x_n\) if \(x_n := (-1)^n + (-1)^n \frac{3^n}{4^{n-2}}\).

(4) Prove that a sequence \((a_n)_{n=1}^{\infty}\) of real numbers that has no Cauchy subsequences must be unbounded.

(5) Suppose \([0, \infty) \xrightarrow{f} \mathbb{R}\) is continuous and satisfies \(\lim_{x \to \infty} f(x) = 0\). Is \(f\) uniformly continuous on \([0, \infty)\)? Why, or why not?

(6) Suppose \(\mathbb{R} \xrightarrow{f} \mathbb{R}\) is uniformly continuous. For each \(n \in \mathbb{N}\), define

\[ f_n(x) := f\left(x + \frac{1}{n}\right). \]

Prove that \((f_n)_{n=1}^{\infty}\) converges uniformly, and find the limit function.

(7) Determine whether or not the following series converges.

\[ 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \cdots \]

(8) Find the interval of convergence for the series \(\sum_{n=1}^{\infty} \frac{n^n}{n!} (x - 2)^n\).
(9) Suppose that \( R \xrightarrow{f} R \) is differentiable at the point \( a \). Prove that
\[
 f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a - h)}{2h}.
\]

(10) Suppose that \( R \xrightarrow{f} R \) is differentiable with \( f'(x) \neq 0 \) for all \( x \in R \). Prove that \( f \) is injective on all of \( R \).

(11) Let \((a_n)\) be an increasing sequence in \((0, 1)\) with limit 1. Define \([0, 1] \xrightarrow{f} R \) by
\[
 f(x) := \begin{cases} 1 & \text{if } x = a_n \text{ for some } n \in N \\ 0 & \text{otherwise} \end{cases}.
\]
Is \( f \) Riemann integrable? Why, or why not?

(12) Let \( R^2 \xrightarrow{f} R \) be defined by
\[
 f(x, y) := \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & \text{when } (x, y) \neq (0, 0) \\ 0 & \text{when } (x, y) = (0, 0) \end{cases}.
\]
Determine where \( f \) is differentiable.

(13) Let \( R^2 \xrightarrow{f} R \) be defined by
\[
 f(x, y) := \begin{cases} \frac{xy^2}{x^2 + y^4} & \text{when } (x, y) \neq (0, 0) \\ 0 & \text{when } (x, y) = (0, 0) \end{cases}.
\]
(a) Let \( u := (a, b) \) with \( a \neq 0 \). Show that the directional derivative \( D_u f(0, 0) \) exists and find its value.
(b) Show that \( f \) is not differentiable at \((0, 0)\). (Hint: Is it continuous there?)

(14) Suppose that \( R^2 \xrightarrow{f} R \) is a function with the property that for all \( x \in R^2 \), \(|f(x)| \leq |x|^2\). Prove that \( f \) is differentiable at the origin.

(15) (a) Let \( R^2 \xrightarrow{f} R \) be defined by \( f(x, y) := x + y \). Prove that \( f \) is differentiable on \( R^2 \) and that for all \((a, b), (x, y) \in R^2\), \( Df(a, b)(x, y) = x + y \).
(b) Suppose \( R^2 \xrightarrow{\varphi} R \) is defined by
\[
 \varphi(x, y) := \int_0^{x+y} g(t) \, dt \quad \text{where } R \xrightarrow{g} R \text{ is continuous}.
\]
Prove that \( \varphi \) is differentiable and find the derivative.

(16) Let \( V \) be a vector space on which an inner product is defined. Define the norm for \( v \in V \) by \(|v| := \sqrt{<v, v>}\). Show that the norm satisfies the triangle inequality \(|v + w| \leq |v| + |w|\) for any \( v, w \in V \).

(17) Let \( A \) be an invertible symmetric operator on a vector space \( V \). Use the inner product definition of a symmetric operator to show that \( A^{-1} \) is also a symmetric operator.

(18) Take \( A \in \text{Mat}_{m \times m}(K) \).
(a) A square matrix \( A \) is nilpotent if \( A^n = 0 \) for some positive integer \( n \). Show that if \( A \) is nilpotent then \( I - A \) is invertible.
(b) Show that if \( A^3 - A + I = 0 \) then \( A \) is invertible.

(19) (a) Let \( T : V \to W \) be a linear mapping between two vector spaces. Show \( T(0) = 0 \).
(b) Let \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear mapping. Suppose
\[
L([3, 1]) = [1, 2] \quad \text{and} \quad L([-1, 0]) = [1, 1].
\]
Compute \( L([1, 0]) \) and \( L([0, 1]) \).
(c) Give an example of a linear mapping that is not injective on its image.

(20) Let \( V \) be a finite-dimensional vector space over \( \mathbb{R} \) or \( \mathbb{C} \) with an inner product. Let \( A \) be a linear map. Show that
\[
\text{Im} A^T = (\ker A)^\perp,
\]
that is, show the image of \( A^T \) is the orthogonal complement of the kernel of \( A \).

(21) Let \( J_{rs} \) be the \( n \times n \) matrix whose \( rs \)-entry is \( c \) and all other entries are 0. Set \( E_{rs} := I + J_{rs} \).
(a) Compute \( \det E_{rs} \). Note there are two distinct cases.
(b) Let \( A \) be an \( n \times n \) matrix. What is the effect of multiplying \( A \) on the left by \( E_{rs} \)? What is the effect of multiplying \( A \) on the right by \( E_{rs} \)?

(22) Compute the determinant of an arbitrary upper-triangular \( n \times n \) matrix \( A \).

(23) Let \( A = (a_{ij}) \in \text{Mat}_{n \times n}(K) \) be such that
\[
\sum_{j=1}^{n} a_{ij} = c, \quad i = 1, \ldots, n
\]
for some \( c \in K \). Show that \( c \) is an eigenvalue for \( A \).

(24) Consider \( A \in \text{Mat}_{2 \times 2}(\mathbb{R}) \). Does \( A \) necessarily have a real eigenvalue? If so, prove it. If not, give a counterexample.

(25) Give a \( 3 \times 3 \) matrix with real entries whose eigenspace is exactly two-dimensional. Find a basis of generalized eigenvectors for your matrix.