Sample Questions for the PhD Preliminary Exam in Statistics and Probability

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Statistics

1. Suppose that $X_1, \ldots, X_n$ are i.i.d. Poisson($\theta$), and $\theta$ has the Gamma($a, b$) prior distribution with mean $ab$, show that, given $(X_1, \ldots, X_n) = (x_1, \ldots, x_n)$, the posterior distribution is Gamma($a + \sum_{j=1}^{n} x_j, \frac{b}{1 + nb}$).

2. Let $X$ and $Y$ be independent exponential random variables, with densities $f_{\lambda}(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}[[x > 0]]$ and $g_{\mu}(y) = \frac{1}{\mu} e^{-\frac{y}{\mu}}[[y > 0]]$, respectively. We observe $Z$ and $W$ with $Z = \min(X, Y)$ and $W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}$.
Find the MLE of $\lambda$ and $\mu$.

3. Suppose that $X \sim$ Beta ($\alpha, \beta$), show that the Fisher information matrix is:

$\mathbf{I}(\alpha, \beta) = \begin{pmatrix} \psi'(\alpha) - \psi'(\alpha + \beta) & -\psi'(\alpha + \beta) \\ -\psi'(\alpha + \beta) & \psi'(\beta) - \psi'(\alpha + \beta) \end{pmatrix},$

where $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$ (the digamma function) and $\psi'(x) = \frac{d}{dx}\psi(x)$ (the trigamma function).

4. Let $X_1, X_2, \ldots, X_n$ denote the incomes of $n$ person chosen at random from a certain population. Suppose that $f(x, \theta) = c^\theta x^{-(1+\theta)}, x > c$,
where $\theta > 1$ and $c > 0$. ($c$ is known)

a. Express mean income $\mu$ in terms of $\theta$.

b. Find the optimal test statistic for testing $H : \mu = \mu_0$ vs. $K : \mu > \mu_0$.

c. Use the central limit theorem to find a normal approximation to critical value of test in part [b].
5. Suppose that $X_1, \ldots, X_n$ are i.i.d. Poisson($\theta$), and $\theta$ has the Gamma($a, b$) prior distribution whose mean is $ab$, show that, for $a > k$, the Bayes estimator under the loss function $L_k(d, \theta) = \frac{(\theta - d)^2}{\theta}$ is given by

$$\delta_k(x) = E(\Theta^{1-k}|x) = \frac{b}{1 + nb} \left( nx + a - k \right),$$

where $x := \frac{1}{n} \sum_{j=1}^n x_j$.

6. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$ with unknown $\mu \in \mathbb{R}$ and unknown $\sigma^2 > 0$. Let $\theta = (\theta_1, \theta_2) = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times \mathbb{R}_+$.

(a) Find the UMVUE $\hat{q}_n(X_1, X_2, \ldots, X_n)$ of $q(\theta) = \frac{\theta_1}{\sqrt{\theta_2}} = \frac{\mu}{\sigma}$.

(b) Find the limiting distribution of $\sqrt{n}(\hat{q}_n(X_1, X_2, \ldots, X_n) - \frac{\mu}{\sigma})$ as $n \to \infty$.

### Probability

1. Suppose $X_1, X_2, \ldots$ are independent identically distributed random variables taking values 1, $\ldots$, $r$ with probabilities $p_1, \ldots, p_r$. Let $L(X_1, \ldots, X_n)$ denote the likelihood of the sequence $X_1, \ldots, X_n$. Prove that

$$-\frac{1}{n} \log L(X_1, \ldots, X_n) \to h$$

where $h = -\sum_{k=1}^r p_k \log p_k$ is Shannon’s entropy.

2. Suppose that $X_1 \leq X_2 \leq \ldots$ and that $X_n \xrightarrow{P} X$. Show that $X_n \to X$ with probability 1.

3. Suppose $X_n$ has density $f_n(x) = 1 + \cos(2\pi nx)$ on interval $[0, 1]$. Show that $X_n \Rightarrow X$ for some $X$.

4. Suppose $X_1, X_2, \ldots$ are independent random variables with distribution $\Pr(X_k = 1) = p_k$ and $\Pr(X_k = 0) = 1 - p_k$. Prove that if $\sum Var(X_k) = \infty$

$$\frac{\sum_{k=1}^n (X_k - p_k)}{\sqrt{\sum_{k=1}^n p_k(1 - p_k)}} \Rightarrow N(0, 1).$$

5. Prove that for $X > 0$

$$E(e^{X}) = 1 + \int_0^\infty e^t \Pr(X > t) \, dt.$$

6. Suppose that $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous. Show that $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ imply $f(X_n, Y_n) \xrightarrow{P} f(X, Y)$. 

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