Sample Questions for the PhD Preliminary Exam in Algebra and Topology

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Algebra

(1) Consider the polynomial \( f(x) = x^6 - 4x^3 + 1 \in \mathbb{Z}[x] \) which you may assume without proof to be irreducible. Let \( K \) be the splitting field of \( F \) over \( \mathbb{Q} \).
   a) Find all the complex roots of \( f \). Show, in particular, that \( f \) has two real roots whose product is \( 1 \).
   b) Let \( \alpha \) be a real root of \( f \). Show that \( K = \mathbb{Q}(\alpha, \omega) \) where \( \omega \) is a primitive cube root of one. Deduce that \( |\text{Gal}(K, \mathbb{Q})| = 12 \).
   c) Show that \( \text{Gal}(K, \mathbb{Q}) \) is a dihedral group.

(2) Let \( K \) be a field with 64 elements and denote by \( \mathbb{F}_2 \) the field with 2 elements.
   a) Find all subfields of \( K \).
   b) How many elements \( \alpha \in K \) are there such that \( \mathbb{F}_2(\alpha) = K \)?
   c) Determine using (b) the number of irreducible polynomials of degree 6 over \( \mathbb{F}_2 \).

(3) Let \( F \) be a field.
   a) Outline the proof of the fact that \( F[x] \) is a PID.
   b) Let \( R = \{ f(x) \in F[x] \mid f'(0) = 0 \} \). Show that \( R \) is not a UFD and find an ideal that is not principal.
   c) Conversely, show that if \( F \) is a field and \( \nu : R \to \mathbb{Z}^+ \) is a surjective map satisfying the properties above then the set
      \[ D = \{ a \in F \mid \nu(a) \geq 0 \} \]
      is a principal ideal domain with a unique non-zero prime ideal.

(4) Let \( k \) be a field of characteristic \( p > 0 \) and let \( 0 \neq c \in k \). Show that the polynomial \( x^p - x - c \) is irreducible if and only if it has no roots in \( k \). Show that this is false if the characteristic of \( k \) is 0.

(5) A field extension \( K \supset F \) is called biquadratic if \( [K : F] = 4 \) and \( K \) is generated over \( F \) by the roots of two irreducible quadratic polynomials. Prove that the extension \( K \supset F \) is biquadratic if and only if it is Galois with Galois group the Klein four group.

(6) Let \( R \) be a principal ideal domain with a unique non-zero prime ideal \( (p) \).
   a) Show that every element of \( R \) can be expressed uniquely in the form \( up^n \) for some non-negative integer \( n \) and unit \( u \).
   b) Let \( \nu : R \to \mathbb{Z}^+ \) be the function given by \( \nu(up^n) = n \). Show that \( \nu \) satisfies
      \[
      \nu(ab) = \nu(a) + \nu(b);
      \nu(a + b) \geq \min(\nu(a), \nu(b));
      \]
   c) Conversely, show that if \( F \) is a field and \( \nu : R \to \mathbb{Z}^+ \) is a surjective map satisfying the properties above then the set
      \[ D = \{ a \in F \mid \nu(a) \geq 0 \} \]
      is a principal ideal domain with a unique non-zero prime ideal.

(7) (a) State and prove Eisenstein’s criterion for the irreducibility of polynomials over \( \mathbb{Z} \).
   (b) Use this result to prove that the polynomial \( [(x + 1)^p - 1]/x \) is irreducible if \( p \) is prime.
(c) Deduce that the cyclotomic polynomial $\Phi_p(x) = 1 + x + x^2 + \cdots + x^{p-1}$ is irreducible if $p$ is prime.

(8) (a) Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$.
(b) Show that its roots are $\pm\sqrt{1 \pm \sqrt{3}}$.
(c) Let $K_1 = \mathbb{Q}(\sqrt{1 + \sqrt{3}})$, $K_2 = \mathbb{Q}(\sqrt{1 - \sqrt{3}})$. Show that $K_1 \neq K_2$ and that $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3})$.
(d) Determine the galois group of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$.

(9) Let $k$ be a field and let $f(x, y) \in k[x, y]$. Prove that if $f(x, x) = 0$, then $f(x, y)$ is divisible by $x - y$. (Hint: use induction on the degree of $f$ as a polynomial in $x$ with coefficients in $k[y]$).

(10) Let $f(x) = x^4 + 5x + 5$.
(a) Find the roots of $f$. What is the Galois group of $f$ over the real numbers $\mathbb{R}$?
(b) Show that $f$ is irreducible over $\mathbb{Q}$.
(c) Show that the splitting field of $f$ has degree 4 over $\mathbb{Q}$ and find the Galois group of $f$ over $\mathbb{Q}$.
Topology

(1) Prove or disprove.
   (a) The product of two quotient maps is a quotient map.
   (b) The product of connected spaces is connected.

(2) Prove that a product space $\prod_{\lambda \in \Lambda} X_{\lambda}$ is contractible if and only if for each $\lambda \in \Lambda$, the space $X_{\lambda}$ is contractible.

(3) Given a topological space $X$, the cone $C(X)$ of the space $X$ is the topological space $X \times [0, 1]/X \times \{0\}$ (i.e. $C(X)$ is the quotient space obtained from $X \times [0, 1]$ by collapsing $X \times \{0\}$ to a point), and the suspension $\Sigma(X)$ of $X$ is the topological space $X \times [0, 1]$, where for $(a, s), (b, t) \in X \times [0, 1]$, $(a, s) \sim (b, t)$ if $s = t$ and either $a = b$, or $t = 0$, or $t = 1$ (i.e. $\Sigma(X)$ is the quotient of $X \times I$ obtained by identifying $X \times \{0\}$ to a single point and $X \times \{1\}$ to another single point).
   (a) Show that $C(X)$ is contractible (thus simply connected).
   (b) Is $\Sigma(X)$ always simply connected? Prove or disprove.

(4) Let $X$ be the complement of two circles $\{x^2 + y^2 = 1; \ z = 1\}$ and $\{x^2 + y^2 = 1; \ z = -1\}$ in $\mathbb{R}^3$. Show that $X$ is path connected and determine the fundamental group $\pi_1(X)$.

(5) Show that there is no one-to-one continuous map from $\mathbb{R}^n \to \mathbb{R}^2$ for $n > 2$.

(6) Let $C$ be the “boundary circle” of the (compact) Möbius band $\mathbb{M}B$. Attach $\mathbb{M}B$ to the “top” of the cylinder $S^1 \times I$ using any homeomorphism $\mathbb{M}B \supset C \to S^1 \times I$. Then attach the torus $T^2 := S^1 \times S^1$ to the “bottom” of the cylinder using any homeomorphism $T^2 \supset S^1 \times \{(1, 0)\} \to S^1 \times \{0\} \subset S^1 \times I$. Let $X$ be the resulting space. Thus $X$ is obtained by first attaching a Möbius band to the top of a cylinder and then attaching a torus to the bottom of the cylinder. Calculate the fundamental group of $X$.

(7) For each integer $m > 2$ and each $n \in \mathbb{N}$, construct a compact connected $m$-manifold whose fundamental group is the free group on $n$ generators. Can you do this if $m = 2$?

(8) (a) Find the universal covering space of the one point union $X := \mathbb{K} \vee S^1$ of the Klein bottle and the cycle.
   (b) Find a covering space $Y \xrightarrow{p} X$ that corresponds to an infinite cyclic subgroup of the fundamental group of $X$.
   (A description of a covering space includes both a definition of the total space as well as a definition of the covering map, and an indication of why the map is a covering projection.)